The Entropy of Linear Cellular Automata with Respect to Any Bernoulli Measure

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This paper deals with the measure-theoretical entropy of a linear cellular automaton (LCA) $T_{f[-l,r]} : \mathbb{Z}_m^{\mathbb{Z}} \to \mathbb{Z}_m^{\mathbb{Z}}$, generated by a bipermutative local rule $f(x_{-l}, \ldots, x_r) = \sum_{i=-l}^{r} a_i x_i \pmod{m}$ ($m \geq 2$ and l, $r \in \mathbb{Z}^+$), with respect to the Bernoulli measure μ_π on $\mathbb{Z}_m^{\mathbb{Z}}$ defined by a probability vector $\pi = (p_0, p_1, \ldots, p_{m-1})$. We prove that the measure entropy of the one-dimensional LCA $T_{f[-l,r]}$ with respect to any Bernoulli measure μ_π is equal to $-(l+r)\sum_{i=0}^{m-1} p_i \log p_i$.

1. Introduction

Cellular automata (CAs), discovered by Ulam and von Neumann, have been systematically studied by Hedlund from a purely mathematical point of view [1]. The study of CAs from the viewpoint of ergodic theory has received remarkable attention in the past few years [2, 3], because CAs have been widely investigated in many disciplines (e.g., mathematics, physics, and computer science). The study of the endomorphisms and the automorphisms of the full shift and its subshifts was begun by Hedlund in [1].

Although linear cellular automata (LCAs) theory and the entropy of these LCAs have developed somewhat independently, there are strong connections between entropy theory and CA theory. The reader is referred to [1, 4] for a definition and some properties of one-dimensional LCAs.

The notion of entropy has been widely studied in a number of disciplines (e.g., computer science, mathematics, physics, chemistry, and information theory) with different purposes. This notion first arose in thermodynamics as a measure of the heat absorbed (or emitted), when external work is done on a system. In probability theory, it constitutes a measure of the uncertainty.

There are several well-known notions of entropy (e.g., measure entropy, topological entropy, directional entropy, and rotational entropy) of the measure-preserving transformation on the probability

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space in ergodic theory [2, 5-9]. It is important to know how these notions are related to each other. For example, recall that by the variational principle, the topological entropy is the supremum of the measure entropies of invariant measures [5]. In information theory, it is known that these measures carry the maximal amount of information of the system.

In [10], Coven and Paul have shown that a CA is onto if and only if it preserves the Bernoulli measure μ_π on $\mathbb{Z}_m^{\mathbb{Z}}$ defined by the uniform probability vector $\left(m^{-1}, \ldots, m^{-1}\right)$. In [1], Hedlund proved that if a CA $T_{f[-l,r]}$ has a right or left permutative rule, then it is onto; or, equivalently, $T_{f[-l,r]}$ preserves the uniform Bernoulli measure μ_π.

In [2], the measure entropy of a LCA is computed with respect to the uniform Bernoulli measure for the case $a_i = 1$, for all $i \in \mathbb{Z}_m$. Applying the results known for the topological entropy of a LCA, it was proved that the uniform Bernoulli measure is the maximal measure for these LCAs. The question of whether or not the maximal measure is unique was also posed. In [11], the author generalized the results obtained in [2] to any LCAs with bipermutative local rules.

In this short paper we answer the questions raised in [2, 11]. Mainly we want to generalize the LCA to an asymmetric LCA and the uniform Bernoulli measure to any Bernoulli measure (but not necessarily uniform). We compute the measure entropy of the one-dimensional LCA $T_{f[-l,r]}$, generated by a bipermutative local rule $f\left(x_{-l}, \ldots, x_r\right) = \sum_{i=-l}^{r} a_i x_i \,(\mathrm{mod}\, m)$, acting on the space of all doubly-infinite sequences with values in a finite ring $\mathbb{Z}_m$, $(m \geq 2$ and $l, r \in \mathbb{Z}^+)$, with respect to a Bernoulli measure on $\mathbb{Z}_m^{\mathbb{Z}}$ defined by any probability vector. We show that the following equation is satisfied:

$$h_{\mu_\pi}\left(T_{f[-l,r]}\right) = -(l+r) \sum_{i=0}^{m-1} p_i \log p_i,$$

where $h_{\mu_\pi}\left(T_{f[-l,r]}\right)$ is the measure entropy of the one-dimensional LCA $T_{f[-l,r]}$ with respect to the Bernoulli measure μ_π on $\mathbb{Z}_m^{\mathbb{Z}}$ defined by the probability vector $\pi = \left(p_0, p_1, \ldots, p_{m-1}\right)$.

2. Preliminaries

Let $\mathbb{Z}_m = \{0,\ 1, \ldots, m-1\}$ $(m \geq 2)$ be a ring of the integers modulo m and $\mathbb{Z}_m^{\mathbb{Z}}$ be the space of all doubly-infinite sequences $x = (x_n)_{n=-\infty}^{\infty} \in \mathbb{Z}_m^{\mathbb{Z}}$ and $x_n \in \mathbb{Z}_m$. A CA can be defined as a homomor-

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phism $\mathbb{Z}_m^{\mathbb{Z}}$ with product topology. The shift $\sigma : \mathbb{Z}_m^{\mathbb{Z}} \to \mathbb{Z}_m^{\mathbb{Z}}$ defined by $(\sigma x)_i = x_{i+1}$ is a homeomorphism of the compact metric space $\mathbb{Z}_m^{\mathbb{Z}}$.

A CA is a continuous map, which commutes with σ, $T : \mathbb{Z}_m^{\mathbb{Z}} \to \mathbb{Z}_m^{\mathbb{Z}}$ defined by $(T x)_i = f(x_{i-l}, \ldots, x_{i+r})$, where $f : \mathbb{Z}_m^{r+l+1} \to \mathbb{Z}_m$ is a given local rule or map. Favati et al. [4] have stated that a local rule f is linear (or additive) if and only if it can be written as

$$f(x_{-l}, \ldots, x_r) = \sum_{i=-l}^{r} a_i\, x_i \,(\operatorname{mod} m), \tag{1}$$

where at least one of the a_i is nonzero. We consider the one-dimensional LCA $T_{f[-l,r]}$ determined by the local rule f:

$$\begin{aligned}(T x) &= (y_n)_{n=-\infty}^{\infty},\\ y_n &= f(x_{n-l}, \ldots, x_{n+r}) = \sum_{i=-l}^{r} a_i\, x_{n+i} \,(\operatorname{mod} m),\end{aligned} \tag{2}$$

where $a_{-l}, \ldots, a_r \in \mathbb{Z}_m$.

We use the notation $T_{f[-l,r]}$ for the LCA map defined in equation (2) to emphasize the local rule f and the numbers $-l$ and r.

The notion of a permutative CA was first introduced by Hedlund in [1]. If the linear local rule $f : \mathbb{Z}_m^{r+l+1} \to \mathbb{Z}_m$ is given in equation (1), then it is permutative in the j^{th} variable if and only if $\gcd(a_j, m) = 1$, where gcd denotes the greatest common divisor. A local rule f is said to be right (respectively, left) permutative if $\gcd(a_r, m) = 1$ (respectively, $\gcd(a_{-l}, m) = 1$). It is said that f is bipermutative if it is both left and right permutative.

3. The Measure Entropy of the One-Dimensional Linear Cellular Automata

In this section we study the measure entropy of the LCA defined in equation (2) with respect to any Bernoulli measure. In order to state our result, we first recall necessary definitions and theorems (see [5] and [12] for details).

Let $\alpha = \{A_1, \ldots, A_n\}$ and $\beta = \{B_1, \ldots, B_m\}$ be two measurable finite partitions of X, then their join is the partition

$$\alpha \vee \beta = \{A_i \cap B_j : i = 1, \ldots, n;\ j = 1, \ldots, m\}.$$

Also, $T^{-1}\alpha$ is the partition $\{T^{-1} A_1, \ldots, T^{-1} A_n\}$.

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Definition 1. Let $(X, \mathcal{B}, \mu, T)$ be a measure-theoretical dynamical system and let α be a measurable partition of X. The partition α is called a *strong generator* if $\vee_{k=0}^{\infty} T^{-k}\alpha = \mathcal{B}$.

Definition 2. Let α be a measurable partition of X. The quantity

$$H_\mu(\alpha) = -\sum_{A\in\alpha} \mu(A)\log\mu(A)$$

is called the *entropy of the partition* α. Let α be a partition with finite entropy, then the quantity

$$h_\mu(\alpha, T) = \lim_{n\to\infty} \frac{1}{n} H_\mu\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha\right)$$

is called the *entropy of* α *with respect to* T. The quantity

$$h_\mu(T) = \sup_{\alpha}\left\{h_\mu(\alpha, T) : \alpha \text{ is a partition with } H_\mu(\alpha) < \infty\right\}$$

is called the *measure entropy* of $(X, \mathcal{B}, \mu, T)$, that is, the entropy of T (with respect to μ).

Theorem 1. ([5], Theorem 4.18). If T is a measure-preserving transformation (but not necessarily invertible) of the probability space $(X, \mathcal{B}, \mu)$ and if $\mathcal{A}$ is a finite subalgebra of $\mathcal{B}$ with $\vee_{k=0}^{\infty} T^{-k}\alpha = \text{B}$, then $h_\mu(T) = h_\mu(\alpha, T)$.

In order to apply entropy theory to a one-dimensional LCA over the ring $\mathbb{Z}_m$ $(m \geq 2)$, we must define the σ-algebra $\mathcal{B}$ and the Bernoulli measure $\mu : \mathcal{B} \to [0, 1]$. In symbolic dynamical systems, it is well known that this σ-algebra $\mathcal{B}$ is generated by thin cylinder sets

$$C =_a \left[j_0, j_1, \ldots, j_s\right]_{a+s} = \left\{x \in \mathbb{Z}_m^{\mathbb{Z}} : x_a = j_0, \ldots, x_{a+s} = j_s\right\},$$

where $j_0, j_1, \ldots, j_s \in \mathbb{Z}_m$.

Let $\left(p_0, p_1, \ldots, p_{m-1}\right)$ be a probability vector. Recall that the Bernoulli measure is defined by

$$\mu_\pi(\text{C}) = \mu_\pi\left({}_a\left[j_0, j_1, \ldots, j_s\right]_{a+s}\right) = p_{j_0} p_{j_1} \cdots p_{j_s},$$

where $p_{j_k} \in \left(p_0, p_1, \ldots, p_{m-1}\right)$ for all $k = 0, \ldots, s$.

See [5, 12] for the properties of the Bernoulli measure.

Let ξ be the zero-time partition of $\mathbb{Z}_m^{\mathbb{Z}}$: $\xi = \left\{{}_0[i] : 0 \leq i < m\right\}$, where ${}_0[i] = \left\{x \in \mathbb{Z}_m^{\mathbb{Z}} : x_0 = i\right\}$ is a cylinder set for all i, $0 \leq i < m$. So, we can state the partition ξ as

$$\xi = \left\{{}_0[0],\ [1], \ldots,{}_0 [m-1]\right\}. \tag{3}$$

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We now give the following lemma.

Lemma 1. ([11], Lemma 3.4). Suppose that $f(x_{-l}, \ldots, x_r) = \sum_{i=-l}^{r} a_i x_i \pmod m$ is a bipermutative local rule and ξ is a partition of $\mathbb{Z}_m^{\mathbb{Z}}$ given by equation (3), then the partition ξ is a strong generator for the one-dimensional LCA generated by local rule f.

Evidently, if the Bernoulli measure is not uniform, then for all measurable cylinder sets C of configurations the equation $\mu_\pi\left(T_{f\,[-l,r]}^{-1}(\mathrm{C})\right) = \mu_\pi(\mathrm{C})$ cannot be satisfied.

For example, let $f\left(x_{-1}, x_0, x_1\right) = x_{-1} + x_0 + x_1 \pmod 2$. Let μ_π be the Bernoulli measure defined by probability vector $\pi = \left(\frac{1}{4}, \frac{3}{4}\right)$. Then we have

$$\begin{aligned} &\mu_\pi\left(T_{f\,[-1,1]}^{-1}\left({}_0[1]\right)\right) = \\ &\quad \mu_\pi({}_{-1}[111] \cup_{-1} [001] \cup_{-1} [010] \cup_{-1} [100]) = \\ &\quad \frac{28}{64} \neq \frac{3}{4} = \mu_\pi\left({}_0[1]\right). \end{aligned}$$

Thus, the CA $T_{f\,[-1,1]}$ is not a measure-preserving transformation with respect to the Bernoulli measure μ_π.

The following theorem is the main result of this paper. Our purpose here is to compute the measure entropy of a bipermutative LCA with respect to an arbitrary Bernoulli measure.

Theorem 2. Let μ_π be a Bernoulli measure defined by the probability vector $\pi = (p_i)$. Assume that $T_{f\,[-l,r]}$ is a LCA with the bipermutative local rule $f(x_{-l}, \ldots, x_r) = \sum_{i=-l}^{r} a_i x_i \pmod m$. Then we get $h_{\mu_\pi}\left(T_{f\,[-l,r]}\right) = -(l+r) \sum_{i=0}^{m-1} p_i \log p_i$.

Proof. By Lemma 1 we can calculate the measure entropy of the one-dimensional LCA by means of the Kolmogorov-Sinai theorem ([5], Theorem 4.18), namely, $h_{\mu_\pi}\left(T_{f\,[-l,r]}\right) = h_{\mu_\pi}\left(T_{f\,[-l,r]}, \xi\right)$. Let ξ be the zero-time partition of $\mathbb{Z}_m^{\mathbb{Z}}$: $\xi = \left\{{}_0[i] : 0 \leq i < m\right\}$, where ${}_0[i] = \left\{x \in \mathbb{Z}_m^{\mathbb{Z}} : x_0 = i\right\}$ is a cylinder set for all i, $0 \leq i < m$. So, we can state the strong partition ξ as

$$\xi = \left\{{}_0[0], {}_0[1], \ldots, {}_0[m-1]\right\}.$$

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From the definition of entropy we have

$$H_{\mu_\pi}(\xi) = -\sum_{i=0}^{m-1} p_i \log p_i.$$

From Theorem 1 we have

$$h_{\mu_\pi}\left(T_{f[-l,r]}\right) = \lim_{n\to\infty} \frac{1}{n} H_{\mu_\pi}\left(\bigvee_{k=0}^{n-1} T_{f[-l,r]}^{-k} \xi\right)$$

$$= \lim_{n\to\infty} \frac{1}{n} H_{\mu_\pi}\left\{\left\{x \in \mathbb{Z}_m^{\mathbb{Z}} : x_{-n\,l} = i_{-n\,l}, \ldots, x_{n\,r} = i_{n\,r}\right\} : \right.$$

$$\left. i_{-n\,l}, \ldots, i_{n\,r} \in \mathbb{Z}_m\right\} = -\lim_{n\to\infty} \frac{1}{n}$$

$$\sum_{i_{-n\,l},\ldots\, i_{n\,l}=0}^{m-1} p_{i_{-n\,l}} p_{i_{-n\,l+1}} \ldots p_{i_{n\,r}} \log p_{i_{-n\,l}} p_{i_{-n\,l+1}} \ldots p_{i_{n\,r}}.$$

It is easy to prove by induction that the sum on the right-hand side is equal to

$$\left((n\,(l+r)+1) \sum_{i=0}^{m-1} p_i \log p_i\right).$$

Thus we have

$$h_{\mu_{\pi P}}\left(T_{f[-l,r]}\right) = -\lim_{n\to\infty} \frac{1}{n}\left((n\,(l+r)+1) \sum_{i=0}^{m-1} p_i \log p_i\right)$$

$$= -(l+r) \sum_{i=0}^{m-1} p_i \log p_i. \square$$

Example 1. Let $f(x_{-2}, \ldots, x_3) = 3\,x_{-2} + 4\,x_{-1} + 2\,x_0 + 3\,x_1 + 6\,x_2 + 5\,x_3 (\mathrm{mod}\, 8)$. Let μ_π be the Bernoulli measure on $\mathbb{Z}_8^{\mathbb{Z}}$ defined by the probability vector

$$\pi = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}\right). \tag{4}$$

Then we have $h_{\mu_\pi}\left(T_{f[-2,3]}\right) = 13.75$.

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A Bernoulli measure on $\mathbb{Z}_m^{\mathbb{Z}}$ is uniform if the measure of any one-dimensional cylinder is equal to $\frac{1}{m}$. From [11] we have the following result.

Remark 1. Let μ_π be the uniform Bernoulli measure on $\mathbb{Z}_m^{\mathbb{Z}}$ and $f\left(x_{-l}, \ldots, x_r\right) = \sum_{i=-l}^{r} a_i\, x_i (\mathrm{mod}\, m)$, where $f\,[-l, r]$ is bipermutative. Then the measure entropy of the one-dimensional LCA $T_{f\,[-l,r]}$ with respect to μ_π is equal to $\log m^{(l+r)}$.

One of the fundamental invariants in ergodic theory is the notion of entropy, both topological and measure-theoretical. Let $h_{\mu_\pi}\left(T_{f\,[-l,r]}\right)$ be the measure entropy of one-dimensional LCA $T_{f\,[-l,r]}$ with respect to μ_π. A probability measure μ_π satisfying $h_{\mu_\pi}\left(T_{f\,[-l,r]}\right) = h_{\text{top}}\left(T_{f\,[-l,r]}\right)$ is said to be the measure of maximal entropy [2, 5, 7, 11], where the quantity $h_{\text{top}}\left(T_{f\,[-l,r]}\right)$ is the topological entropy of the LCA $T_{f\,[-l,r]}$. In information theory, it is known that these measures carry the maximal amount of information of the system. From Example 1, if the measure μ_π is chosen as the uniform Bernoulli measure, then $h_{\mu_\pi}\left(T_{f\,[-2,3]}\right) = h_{\text{top}}\left(T_{f\,[-2,3]}\right) = 15$ (see [2, 9, 11]). Therefore, we conclude that the value 15 is the maximal amount of information that can be carried by the uniform Bernoulli measure μ_π. But, the amount of information carried by the Bernoulli measure defined by the probability vector in equation (4) is 13.75 bits.

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