# Modeling Rope Reliability

**Bolesław Kopocinski**

*Mathematical Institute, University of Wrocław*
*pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland*

In this paper we attempt to model the reliability of a steel rope that is susceptible to breakage during its operation. Each individual strand of the rope is modeled using a counting process for breakages and the associated process describing new breakages. The decision to replace the rope is based on the maximum number of breakages in the strands along a given length. Using a stochastic simulation, significant simplifications in the model of the associated process are introduced. The decision to replace the rope is based on the convolution of the associated processes acting on the strands. This is approximated by a Gaussian process. Deterministic modeling of the strands indicates the possibility of estimating the parameters of the processes that characterize the rope.

## 1. Description of the Rope

J. Czaplicki [1] suggests the following mathematical description of a steel rope. The rope is coiled from $N$ strands and it is $K$ screw threads long. The strands are susceptible to breakdowns (fracture) during operation. If a strand breaks down, the damage extends over an interval of $m \geq 2$ screw threads symmetrically around the breakage point. A new breakdown occurs at random over the whole strand excluding the damaged segment. We also assume that the rate of strand breakdowns does not increase as a result of placing a larger weight on the functional strands. The ends of the rope are attached, which eliminates the problem of modeling the rope ends. The rope is out of order if $s^{**}$ strand breakages appear within a space of $m$ threads. Examples of real rope parameters are suggested in Section 4.

## 2. Mathematical Model of the Rope

The main objects in our considerations are the parallel strands from which the rope is rolled. One strand of the rope is modeled as an interval $[0, K]$ of length $K$ on which a Poisson process $\{s(d), 0 \leq d \leq K\}$ of a given rate occurs. This rate may be dependent upon the time for which a rope has operated. This possibility is not examined here, although the dependence of the rope reliability upon the time for


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which a rope has operated may be of interest and practical importance.

We define the associated binary process $\{a(d), 0 \leq d \leq K\}$ on $[0, K]$. It is assumed that the value 1 describes a damaged site and the value 0 describes a functional site.

It is well known that the signals of a Poisson stream $\{s(d), 0 \leq d \leq K\}$ conditioned by $s(K) = s$ are uniformly distributed on $[0, K]$. Under this condition these signals can be numbered and the associated process can be defined inductively according to the appearance of successive signals. If the $i^{\text{th}}$ signal, which appears at $d_i$, does not occur in an already damaged site, that is, previously $a(d_i) = 0$, then we set $a(d) = 1$ for $d_i - \frac{m}{2} \leq d \leq d_i + \frac{m}{2}$. We declare that the rope is circular in shape with no endpoints, so the damage that does not fit on one end spreads to the other end of the strand. If the signal occurs at a place where a blockade already exists, for example, $a(d_i) = 1$, then no changes in the state of the associated process are made.

Denote the counting process for the signals that generate damage by $\{s^*(d),\ 0 \leq d \leq K\}$. The number $s^*(K) = s^*$ given that $s(K) = s$ is obviously also a random variable.

Let us consider $N$ independent probabilistic copies of $s_j(d)$, $a_j(d)$, and $s_j^*(d)$, $1 \leq j \leq N$ of the processes describing the $N$ strands that form the rope. The process $S(d)$ is the total number of signals of the Poisson processes, $A(d)$ is the total number of blockades, and $S^*(d)$ is the number of breakdowns in the rope. They are defined by:

$$S(d) = \sum_{j=1}^{N} s_j(d) \qquad A(d) = \sum_{j=1}^{N} a_j(d),$$

$$S^*(d) = \sum_{j=1}^{N} s_j^*(d) \qquad 0 \leq d \leq K.$$

## 3. Problem

Observing the process $S^*(d)$ enables us to decide when to replace the rope, but the extreme congestion of breakdowns is taken into consideration in practice. The decision criterion is usually the maximum number of breakdowns in $m$ threads of the rope. Let us define the random variable

$$S^{**} = \max_{0 \leq d \leq K} (S^*(d+m) - S^*(d)),$$


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where for $d+m>K$ we substitute the argument $d+m-K$, since the ends of the rope are attached. Note that $S^*(d+m)-S^*(d)=A\left(d+\frac{1}{2}m\right)$. In fact, let $d\in[0,K]$, and let an event of the Poisson process $\{s_j(d),\ 0\le d\le K\}$ take place in $x_i$ generating the damage in a process $\{a_j(d),\ 0\le d\le K\}$ where $d-\frac{1}{2}m<x_j\le d-\frac{1}{2}m$, as well as $a_j\left(d+\frac{1}{2}=1\right)$. Because $S^*(d+m)-S^*(d)$ denotes the total number of breakdowns in $(d,d+m]$ it is equal to $A\left(d+\frac{1}{2}m\right)$. Thus the decision variable is of the form

$$S^{**}=\max_{\frac{1}{2}m\le d\le K+\frac{1}{2}m} A(d)=\max_{0\le d\le K} A(d).$$

Note that given a small number of signals has occurred in the associated process $a(d)$, the length of a blockade is equal to $m$ with a large probability. Given that a larger number of signals have occurred, blockades can become extended and the lengths of neighboring functional and damaged intervals are mutually dependent. The central limit theorem suggests that $A(d)$, which depends on $N$, if $N\to\infty$ is approximately a stationary Gaussian process (see [2], Theorem and Comments). In analyzing the limiting properties of the extreme of this process we limit our considerations to stationary processes. Such a process is characterized by its expected value, variance, and correlation function.

## 4. Stochastic Simulation of the Rope

The random variable $S^{**}$ defined previously is rather difficult to analyze. Sensible simplifications can be proposed after a preliminary investigation of the rope characteristics using stochastic simulations. Our simulations enable observing the processes $a(d)$ and $s^*(d)$ to describe a strand and the processes $A(d)$ and $S^*(d)$ to describe the rope as a whole.

In these simulations we divide the interval $[0,K]$ into units of length equal to one thread of the screw. For example, a blockade on the interval $[0,m]$ is defined to cover the set $\{1,2,\ldots,m\}$. It is convenient to assume that $m$ is odd. The center of a blockade segment is integer-valued and the blockade covers $\frac{1}{2}(m-1)$ units to both the left and right of the center. The following values for the rope parameters are considered:

- $K=1600$ is the length of the rope.
- $N=20$ is the number of strands in the rope (hence, the total length of the strands in the rope is equal to $NK=32\,000$).


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- $m = 11$ is the blockade length.
- $s = 100, 200, 400, 800$ gives values taken for the number of breakages.

Using these parameters, we simulate the associated processes $a_j(d)$, $1 \leq j \leq N$, as independent realizations of the appropriate stationary processes. We analyze the number of breakages, as well as the lengths of functional and damaged intervals. We calculate the correlation between the lengths of neighboring functional and damaged intervals and the correlation function of the process. The extreme $S^{**}$ is also observed. The simulation was repeated 20 times for each case.

We introduce the following notation: $X$ is the length of a damaged interval, $Y$ is the length of a functional interval, $S^* = S^*(K)$ is the number of signals generating damage, $C$ is the number of blockades in the associated process, $C_2$ is the number of blockades in the associated process for which $m < X \leq 2m$ (two breakages in a blockade), $C_3$ is the number of cycles in the associated process for which $X > 2m$ (at least three breakages in a blockade), and $c(\tau) = \mathrm{Corr}(a(d), a(d+\tau))$ is the correlation function of the process $a(d)$. Let $X_2 = X \,|\, (m < X \leq 2m)$, $X_3 = X \,|\, (X > 2m)$. In the case considered, the correlation function of the process $A(d)$ is identical to the correlation function of $a(d)$.

## 5. Conclusions from the Stochastic Simulations

Table 1 gives the results of the stochastic simulations: $S^*$, $C$, $C_2$, and $C_3$ describe the number of blockades in the associated process and the parameters of the probability distribution of the length of damaged and functional intervals. Table 1 also gives the characteristics of damaged intervals of selected lengths, which are useful when describing the structure of this random variable, its expected value, and the variance of the process $A(d)$. Table 2 gives the correlation function of the associated process. These are the means of the results from the 20 repetitions of the simulation.

Note that given there is a small number of blockades in the associated process, the expected value of the length of a functional interval is almost equal to its standard deviation. Hence, we can assume that the distribution of this length is approximately exponential. The correlation between the lengths of neighboring damaged and functional intervals is close to zero, therefore it can be assumed that these random variables are independent. We consider the following simplified associated process.


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| | | | | | |
|---|---|---|---|---|---|
| | Number of breakages | | | | |
| 1.1 | $s$ | 100 | 200 | 400 | 800 |
| 1.2 | $S^*$ | 98.8 | 192.8 | 374.5 | 701.6 |
| 1.3 | $C$ | 96.9 | 185.4 | 346.3 | 604.8 |
| 1.4 | $C_2$ | 1.9 | 6.8 | 24.3 | 72.7 |
| 1.5 | $C_3$ | 0 | 0.3 | 1.9 | 11.1 |
| | Length of functional intervals | | | | |
| 1.6 | $E(Y)$ | 315.6 | 160.0 | 80.5 | 40.5 |
| 1.7 | $D(Y)$ | 312.2 | 156.0 | 78.3 | 39.5 |
| | Length of damaged intervals | | | | |
| 1.8 | $E(X)$ | 11.17 | 11.34 | 11.69 | 12.36 |
| 1.9 | $D(X)$ | 1.23 | 1.78 | 2.57 | 3.70 |
| 1.10 | $c(X, Y)$ | −0.001 | 0.002 | 0.015 | 0.024 |
| | Conditioned length of damaged intervals | | | | |
| 1.11 | $P(m < X \leq 2m)$ | 0.020 | 0.037 | 0.070 | 0.120 |
| 1.12 | $E(X_2)$ | 19.7 | 19.6 | 19.5 | 19.5 |
| 1.13 | $D(X_2)$ | 1.84 | 1.75 | 1.76 | 1.75 |
| 1.14 | $P(X > 2m)$ | 0 | 0.002 | 0.005 | 0.018 |
| 1.15 | $E(X_3)$ | - | 27.7 | 28.8 | 29.3 |
| 1.16 | $D(X_3)$ | - | 2.21 | 3.30 | 4.48 |
| 1.17 | $E(A(t))$ | 0.68 | 1.32 | 2.53 | 4.67 |
| 1.18 | $D(A(t))$ | 0.81 | 1.10 | 1.48 | 1.90 |
| | Extreme value | | | | |
| 1.19 | $E(S^{**})$ | 4.1 | 5.6 | 7.8 | 11.15 |

**Table 1**. Parameters of the associated process $a(d)$.

The results concerning the damaged intervals suggest the following structure for $X$. $X = m$ with some probability $p_1$ (one breakdown in a blockade) or $X = m + X_1$ with probability $p_2 = 1 - p_1$ (at least two breakdowns in a blockade). For small $p_2$, $X_1 = X_0 + \frac{1}{2}(m-1)$, where $X_0$ is uniformly distributed on $\left\{1, 2, \ldots, \frac{1}{2}(m+1)\right\}$. The extension of a blockade happens if a new breakdown occurs sufficiently close to an existing blockade. Such an extension can occur again, but for small $p_2$ such consecutive extensions may be neglected. Thus, we have

$$X = m + \delta_1 X_1,$$


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| $s$ | 100 | 200 | 400 | 800 |
|---|---|---|---|---|
| $\tau$ | Correlation function $c(\tau)$ | | | |
| 1 | 0.91 | 0.91 | 0.90 | 0.90 |
| 2 | 0.81 | 0.81 | 0.81 | 0.79 |
| 3 | 0.72 | 0.72 | 0.71 | 0.69 |
| 4 | 0.63 | 0.63 | 0.62 | 0.59 |
| 5 | 0.53 | 0.53 | 0.52 | 0.50 |
| 6 | 0.44 | 0.44 | 0.43 | 0.40 |
| 7 | 0.35 | 0.35 | 0.34 | 0.31 |
| 8 | 0.26 | 0.26 | 0.26 | 0.22 |
| 9 | 0.16 | 0.17 | 0.17 | 0.13 |
| 10 | 0.07 | 0.07 | 0.08 | 0.05 |
| 11 | −0.03 | −0.02 | −0.01 | −0.04 |

**Table 2.** Correlation function $c(\tau)$ of the process $A(d)$.

where $\delta_1$ denotes the binary random variable $P(\delta_1 = 0) = p_1$, $P(\delta_1 = 1) = p_2$. It is easy to check that $\mathrm{E}(X_0) = m_0 = \frac{1}{4}(m+3)$, $\mathrm{E}(X_1) = m_1 = \frac{1}{4}(3m+1)$, $\mathrm{D}^2(X_1) = \sigma_1^2 = \frac{1}{12}\left(\frac{1}{4}(m+1)^2 - 1\right)$. Hence, $\mathrm{E}(X) = m + p_2 m_1$, $\mathrm{D}^2(X) = p_2 \sigma_1^2 + p_1 p_2 m_1^2$.

If $p_2$ is somewhat larger, then we may assume that the probability of three extensions is negligible. In this case we assume

$$X = m + \delta_1 X_1 + \delta_1 \delta_2 X_2,$$

where $\delta_1$, $\delta_2$, $X_1$, and $X_2$ are mutually independent. The $X_i$ have the probability distribution function as described earlier and the $\delta_i$ are binary random variables. It can be shown that

$$\mathrm{E}(X) = m + \left(p_2 + p_2^2\right) m_1,$$
$$\mathrm{D}^2(X) = \left(p_2 + p_2^2\right)\sigma_1^2 + p_1 p_2 \left(1 + 3 p_2 + p_2^2\right) m_1^2.$$

Table 1 (see lines (1.8) and (1.9)) confirms the reasonableness of this approximation: when $s = 100$, the possibility of two extensions may be neglected, when $s = 200, 400, 800$, the possibility of three extensions may be neglected.

## 6. Deterministic Simulation of the Rope

Deterministic simulations can be carried out by replacing the random variables used in the stochastic simulation by their expected values. Using this approach to model the strands, we introduce the following notation. As earlier, $s$ is the number of signals in the Poisson process,


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$i$ is the index of a signal, $s_i^*$ is the number of the signals counted, $c_i$ is the number of cycles of the associated process, $x_i$ is the total length of a blockade, $y_i$ is the total length of a functional interval, $\pi_i$ is the number of signals counted, $\pi_i^*$ is the number of blockades of length $m$, and $\pi_i - \pi_i^*$ is the number of extensions to blockades.

We wish to derive recurrence formulas for the expected length of the damaged intervals of strands. When $s$ is small (in comparison to $K$), we make the following simplifications:

- The quantities $s_i^*$, $c_i$, $x_i$, $y_i$, $\pi_i$, and $\pi_i^*$ are deterministic.
- The length of each functional interval is at least $m$.
- The length of each extension to a blockade is constant, that is, it is equal to $m_1 = \frac{1}{4}(3\,m + 1)$.

We assume that the following recurrence formulas hold:

$$\begin{aligned}
&s_1^* = 1, \quad c_1 = 1, \quad x_1 = m, \quad y_1 = K - m,\\
&\pi_{i+1} = \frac{y_i}{K}, \quad s_{i+1}^* = s_i^* + \pi_{i+1},\\
&\pi_{i+1}^* = \frac{1}{K}\left(y_i - 2\left(m_0 + 1\right)c_i\right), \quad c_{i+1} = c_i + \pi_{i+1}^*,\\
&x_{i+1} = x_i + m\,\pi_{i+1}^* + (m_1 + 1)\left(\pi_{i+1} - \pi_{i+1}^*\right),\\
&y_{i+1} = K - x_{i+1}, \quad 1 \le i \le s - 1.
\end{aligned}$$

The quantities $m_0$ and $m_1$ are increased by 1 due to the discretization of the modeled strands.

Table 3 gives the results of the deterministic simulation. They are consistent with the results of the stochastic simulations presented in Tables 1 and 2; see lines (1.2) and (3.2), (1.3) and (3.3), (1.6) and (3.4), (1.8) and (3.5), and (1.17) and (3.6).

| | | | | | |
|---|---|---|---|---|---|
| 3.1 | $s$ | 100 | 200 | 400 | 800 |
| 3.2 | $s_s^*$ | 98.3 | 193.3 | 373.9 | 700.5 |
| 3.3 | $c_s$ | 97.0 | 188.0 | 353.2 | 624.1 |
| 3.4 | $y_s / c_s$ | 318.9 | 159.0 | 79.0 | 39.1 |
| 3.5 | $x_s / c_s$ | 11.13 | 11.27 | 11.56 | 12.16 |
| 3.6 | $N\,x_s / K$ | 0.67 | 1.32 | 2.55 | 4.74 |

**Table 3.** Results of the deterministic simulation.


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## 7. The Extreme Value

Numerous texts have been devoted to analyzing the probability distribution of the extreme of a Gaussian process [3-5]. We are interested in the asymptotic properties of $P(S^{**} > s^{**})$. In the case of standardized Gaussian processes, the probability distribution function of the extreme value depends upon the shape of the correlation function at zero. Assuming that the lengths of blockades and functional intervals have an exponential probability distribution, it is well known (see [6, p. 295]) that the correlation function has an exponential shape. The stochastic simulation shows that in our problem the correlation function is linear in the neighborhood of zero. A proof that the extreme of $A(d)$ has a double exponential distribution as $K \to \infty$ can be found in [4] (see Theorem 8.2.7).

More precisely, let $p = \mathrm{E}(a(d))$ and define the following standardized process and its extreme:

$$\Xi(d) = \frac{A(d) - N p}{\sqrt{N p (1-p)}}, \quad 0 \le d \le K, \quad \Xi_K^* = \sup_{0 \le d \le K} \Xi(d).$$

The following result holds

$$\lim_{K \to \infty} P(\alpha_K (\Xi_K^* - \alpha_K) \le x) = \Psi(x) = \exp(-e^{-x}),$$

where $\alpha_K = \sqrt{2 \log K}$.

This result enables the specification of quantile intervals for the extreme value. For some given $\alpha$, let $u_\alpha$ be the solution of the equation $\Psi(u) = \alpha$. Using the continuity correction for a discrete variable, a quantile interval for $\Xi^*$ at the level of $1 - 2\alpha$ is given by

$$S_{\min}^{**} = N p - \frac{1}{2} - \left( \frac{u_\alpha}{\alpha_K} + \alpha_K \right) \sqrt{N p (1-p)} \le S^{**} \le S_{\max}^{**} =$$

$$N p + \frac{1}{2} + \left( \frac{u_{1-\alpha}}{\alpha_K} + \alpha_K \right) \sqrt{N p (1-p)}.$$

## 8. Example

Let us consider the numerical values $N p = \mathrm{E}(A(d))$ presented in Table 1. Line (1.17) characterizes the deterministic simulation of $A(d)$. Table 4 gives quantile intervals for the extreme value at the level of $1 - 2\alpha = 0.95$. The final column gives the range of the extreme values observed in the stochastic simulations. These results fit the quantile intervals well.


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| $s$ | $S^{**}_{\min}$ | $S^{**}_{\max}$ | Range in 20 stochastic simulations |
|---|---|---|---|
| 100 | 3.02 | 5.07 | 3 - 5 |
| 200 | 4.71 | 7.15 | 5 - 7 |
| 400 | 7.24 | 10.16 | 7 - 9 |
| 800 | 10.81 | 14.27 | 10 - 12 |

**Table 4.** Quantile intervals for $S^{**}$ at the level of $1 - 2\alpha = 0.95$.

## 9. Conclusions

The reliability of a rope can be considered with respect to the number of breakages in its strands during operation. The state of the strands may be described by an associated process. The state of the rope is described by the convolution of these associated processes, which can be approximated using a stationary Gaussian process. The parameters of this process depend on the number of breakages in the strands. The correlation function of this process has the desired shape in the neighborhood of zero. The theorem on the limit of the extreme of the Gaussian process gives the asymptotic distribution of the extreme number of breakages along an interval of $m$ threads of the rope, $S^{**}$. The decision on whether to replace the rope may be undertaken on the basis of $S^{**}$.

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