2. Compare the n^2 entries A_{ij} and B_{ij}. This step requires at most n^2 comparisons. (This maximal value is needed in the case that f is a fixed point system.)
3. f is a fixed point system if and only if the matrices A and B are equal.

It is well known that matrix multiplication requires $2n^3 - n^2$ addition or multiplication operations. Since $t + 1 < n\log_2(q) + 2$, the number of operations required in step 1 is bounded above by $\left(2n^3 - n^2\right)\left(n\log_2(q) + 2\right)$. Summarizing, we have the following upper bound $N(n, q)$ for the number of operations in steps 1 and 2:

$$N(n, q) := \left(2n^3 - n^2\right)\left(n\log_2(q) + 2\right) + n^2.$$

For a fixed size q of the finite field $\mathbf{F}_q$ used it holds that

$$\lim_{n\to\infty} \frac{N(n, q)}{n^4} = 2\log_2(q)$$

and we can conclude $N(n, q) \in O\left(n^4\right)$ for a fixed q. The asymptotic behavior for a growing number of variables and growing number of field elements is described by

$$\lim_{\substack{n\to\infty \\ q\to\infty}} \frac{N(n, q)}{n^4\log_2(q)} = 2.$$

Thus, $N(n, q) \in O\left(n^4\log_2(q)\right)$ for $n, q \to \infty$.

It is necessary to comment on the arithmetic operations performed during the matrix multiplications. Since the matrices are elements of the matrix monoid $M\left(n\times n; E_q\right)$, the arithmetic operations are operations in the monoid E_q. The addition (resp., multiplication) operation on E_q requires an integer number addition (resp., multiplication) and a reduction using red_q (see Section 2). For a detailed description of integer number representation and arithmetic in typical computer algebra systems see [28, Chapter 4].

The reduction $\mathrm{red}_q(a)$ of an integer number $a \in \mathbb{N}_0$, $a \geq q$ is obtained as the degree of the remainder of the polynomial division $\tau^a \div (\tau^q - \tau)$. According to [28, Section 4.6.5] this division requires $O(2\,(\deg(\tau^a) - \deg(\tau^q - \tau))) = O(2\,(a - q))$ integer number operations. However, we know that the reductions $\mathrm{red}_q(.)$ are applied to the result of (regular integer) addition or multiplication of elements of E_q and therefore

https://doi.org/10.25088/ComplexSystems.18.3.307

$$a - q \leq \begin{cases} 2(q-1) - q = q - 2 \\ (q-1)^2 - q = q^2 - q + 1. \end{cases}$$

As a consequence, in the worst case scenario, one addition (resp., multiplication) in the monoid E_q requires $O(q)$ (resp., $O(q^2)$) regular integer number operations.

Since E_q is a finite set and only the results of n^2 pairwise additions and n^2 pairwise multiplications are needed, the numbers are stored in a table after the first time they are calculated while the algorithm is running. Since most of the commercial and freely available computer algebra systems provide implementations of finite field arithmetic and arithmetic of polynomials over a finite ring, the implementation of our algorithm is a very simple task. We have successfully implemented the algorithm in Maple$^{\text{TM}}$. The code can be made available from the author upon request.

Since the matrix multiplications dominate the complexity of the algorithm, for very large systems more efficient matrix multiplication algorithms could be used. Indeed, using Strassen's algorithm [29, 30], a complexity of $O(n^{\omega+1}\log_2(q))$, where $\omega \leq \log_2(7) \approx 2.807$ could be reached. The performance could be substantially improved through the use of parallelization techniques [31, 32].

It is pertinent to mention that while this article was being peer-reviewed, the article [14] was published, in which an algorithm of complexity $O(n^3 \log_2(n \log_2(q))$ is presented, that is used to determine whether an n-dimensional linear dynamical system $L : R^n \to R^n$ over a finite ring R of cardinality $q = |R|$ is a fixed point system. While the authors of [14] did perform a complexity analysis of their algorithm, they did not elaborate on the details of a concrete computer implementation of the arithmetic operations on the finite ring R and thus, did not provide information on the computational cost of such operations.

In [13] it is stated that in order to exploit their results algorithmically, an algorithm would be required to determine whether or not a linear dynamical system $L : R^n \to R^n$ over a finite ring R is a fixed point system. The algorithm presented in [14] provides this missing ingredient. Nevertheless, for a given $f \in MF_n^n(\mathbf{F}_q)$, prior to being able to apply the algorithm from [14], a linear dynamical system $L(f) : (\mathbb{Z}/(q-1))^n \to (\mathbb{Z}/(q-1))^n$ and a Boolean monomial dynamical system $T(f) \in MF_n^n(\mathbf{F}_2)$ need to be constructed (see [13] for the details). The overall computational complexity of an algorithm exploiting the results of [13] and [14] still remains to be investigated.

https://doi.org/10.25088/ComplexSystems.18.3.307

Acknowledgments

We would like to thank Dr. Omar Colón-Reyes for his hospitality and a very fruitful academic interaction at the University of Puerto Rico, Mayagüez. We are grateful to Dr. Bodo Pareigis for offering the opportunity to participate in a great seminar at the Ludwig-Maximilians-Universität in Munich, Germany. We also would like to express our gratitude to Dr. Michael Shapiro for very helpful comments. We are grateful to Dr. Karen Duca for her support and to Dr. David Thorley-Lawson for providing an excellent research environment at the Pathology Department of Tufts University, the institute where the material for this paper was conceived. Moreover, we thank Dr. Jill Roughan, Dr. Michael Shapiro and Dr. David Thorley-Lawson for proofreading the manuscript.

The author acknowledges support by a Public Health Research Grant (RO1 AI062989) to Dr. David Thorley-Lawson at Tufts University, Boston, MA, USA. The author was affiliated with Centre for Mathematical Sciences, Munich University of Technology, Boltzmannstr.3, 85747 Garching, Germany and also with Pathology Department, Tufts University, 150 Harrison Ave., Boston, MA 02111, USA during the development of the material presented here.

References

[1] A. W. Burks, ed., *Essays on Cellular Automata*, Urbana, IL: University of Illinois Press, 1970.

[2] J. Reger and K. Schmidt, "Modeling and Analyzing Finite State Automata in the Finite Field F_2," *Mathematics and Computers in Simulation*, **66**(2-3), 2004 pp. 193-206. doi .1016/j.matcom.2003.11.005.

[3] C. L. Barrett, H. S. Mortveit, and C. M. Reidys, "Elements of a Theory of Simulation II: Sequential Dynamical Systems," *Applied Mathematics and Computation*, **107**(2-3), 2000 pp. 121-136. doi.10.1016/S0096-3003(98)10114-5.

[4] R. Lidl and H. Niederreiter, *Finite Fields*, 2nd ed., Vol. 20 (*Encyclopedia of Mathematics and Its Applications*), New York: Cambridge University Press, 1997 (foreword by P. M. Cohn).

[5] E. Delgado-Eckert, "Monomial Dynamical and Control Systems over a Finite Field and Applications to Agent-Based Models in Immunology," Ph.D. thesis, Munich, Germany: Technische Universität München, 2008. http://mediatum2.ub.tum.de/doc/645326/document.pdf.

[6] J. Kari, "Theory of Cellular Automata: A Survey," *Theoretical Computer Science*, **334**(1-3), 2005 pp. 3-33. doi.10.1016/j.tcs.2004.11.021.

[7] R. Bartlett and M. Garzon, "Monomial Cellular Automata," *Complex Systems*, 7(5), 1993 pp. 367-388.

[8] O. Colón-Reyes, R. Laubenbacher, and B. Pareigis, "Boolean Monomial Dynamical Systems," *Annals of Combinatorics*, 8(4), 2004 pp. 425-439. doi.10.1007/s00026-004-0230-6.

https://doi.org/10.25088/ComplexSystems.18.3.307

[9] A. Khrennikov and M. Nilsson, "On the Number of Cycles of p-adic Dynamical Systems," *Journal of Number Theory*, **90**(2), 2001 pp. 255-264. doi.10.1006/jnth.2001.2665.

[10] M. Nilsson, "Fuzzy Cycles of p-adic Monomial Dynamical Systems," *Far East Journal of Dynamical Systems*, **5**(2), 2003 pp. 149-173.

[11] T. Vasiga and J. Shallit, "On the Iteration of Certain Quadratic Maps over $GF(p)$," *Discrete Mathematics*, **277**(1-3), 2004 pp. 219-240.

[12] O. Colón-Reyes, *Monomial Dynamical Systems over Finite Fields*, Ph.D. thesis, Blacksburg, VA: Virginia Tech., 2005.

[13] O. Colón-Reyes, A. S. Jarrah, R. Laubenbacher, and B. Sturmfels, "Monomial Dynamical Systems over Finite Fields," *Complex Systems*, **16**(4), 2006 pp. 333-342.

[14] G. Xu and Y. M. Zou, "Linear Dynamical Systems over Finite Rings," *Journal of Algebra*, **321**(8), 2009 pp. 2149-2155.

[15] E. Delgado-Eckert, "Boolean Monomial Control Systems," *Mathematical and Computer Modelling of Dynamical Systems*, **15**(2), 2009 pp. 107-137. doi.10.1080/13873950902808594.

[16] B. Elspas, "The Theory of Autonomous Linear Sequential Networks," *IRE Transactions on Circuit Theory*, **6**(1), 1959 pp. 45-60.

[17] R. A. Hernández Toledo, "Linear Finite Dynamical Systems," *Communications in Algebra*, **33**(9), 2005 pp. 2977-2989. doi.10.1081/AGB-200066211.

[18] J. Reger and K. Schmidt, "A Finite Field Framework for Modeling, Analysis and Control of Finite State Automata," *Mathematical and Computer Modelling of Dynamical Systems (MCMDS)*, **10**(3-4), 2004 pp. 253-285. doi.10.1080/13873950412331300142.

[19] D. K. Milligan and M. J. D. Wilson, "The Behavior of Affine Boolean Sequential Networks," *Connection Science*, **5**(2), 1993 pp. 153-167.

[20] P. Cull, "Linear Analysis of Switching Nets," *Biological Cybernetics*, **8**(1), 1971 pp. 31-39. doi.10.1007/BF00270831.

[21] W. Just, "The Steady State System Problem Is NP-Hard Even for Monotone Quadratic Boolean Dynamical Systems." (Jun 3, 2006) www.math.ohiou.edu/~just/PAPERS/monNPh14.pdf.

[22] F. Harary, R. Z. Norman, and D. Cartwright, *Structural Models: An Introduction to the Theory of Directed Graphs*, New York: Wiley & Sons, 1965.

[23] V. Pták and I. Sedlaček, "On the Index of Imprimitivity of Nonnegative Matrices," *Czechoslovak Mathematical Journal*, **8**(83), 1958 pp. 496-501.

[24] E. V. Denardo, "Periods of Connected Networks and Powers of Nonnegative Matrices," *Mathematics of Operations Research*, **2**(1), 1977 pp. 20-24. doi.10.1287/moor.2.1.20.

[25] R. A. Brualdi and H. J. Ryser, *Combinatorial Matrix Theory*, Vol. 39 (*Encyclopedia of Mathematics and Its Applications*), New York: Cambridge University Press, 1991.

[26] P. Lancaster and M. Tismenetsky, *The Theory of Matrices: With Applications*, 2nd ed., *(Computer Science and Applied Mathematics)*. Orlando, FL: Academic Press Inc., 1985.

https://doi.org/10.25088/ComplexSystems.18.3.307

[27] B. De Schutter and B. De Moor, "On the Sequence of Consecutive Powers of a Matrix in a Boolean Algebra," *SIAM Journal on Matrix Analysis and Applications*, **21**(1), 1999 pp. 328-354. doi.10.1137/S0895479897326079.

[28] M. Kaplan, *Computeralgebra*, Berlin: Springer-Verlag, 2004.

[29] V. Strassen, "Gaussian Elimination Is Not Optimal," *Numerische Mathematik*, **13**(4), 1969 pp. 354-356.

[30] D. H. Bailey, K. Lee, and H. Simon, "Using Strassen's Algorithm to Accelerate the Solution of Linear Systems," *The Journal of Supercomputing*, **4**(4), 1991 pp. 357-371. doi .10.1007/BF00129836.

[31] M. Ben-Ari, *Principles of Concurrent and Distributed Programming: Algorithms and Models*, 2nd ed., New York: Addison-Wesley, 2006.

[32] A. Pollard, D. J. K. Mewhort, and D. F. Weaver, eds., *High Performance Computing Systems and Applications*, New York: Springer-Verlag, 2000.

https://doi.org/10.25088/ComplexSystems.18.3.307