Methodological Approaches for the Fokker–Planck Equation Associated to Nonlinear Stochastic Differential Systems with Uncertain Parameters

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This paper is an extension of work originally presented at the World Conference on Complex Systems. In this paper, methodological approaches and numerical procedures are elaborated for nonlinear stochastic differential equations with uncertain parameters. The associated Fokker–Planck equation is used to get the distribution function. Mathematical developments based on the meshfree method with radial basis functions and on exponential closure combined with Monte Carlo and conditional expectation methods are elaborated for numerical solutions. The obtained approximate solutions compare well with available solutions and the effectiveness and accuracy of the proposed methods are demonstrated.

Keywords: exponential closure method; meshfree method; radial basis function; Fokker–Planck equation; stochastic differential equation; uncertain parameters; probability density function; Monte Carlo; conditional expectation

1. Introduction

The Fokker–Planck equation was first used by Adriaan Fokker and Max Planck to describe the probability density function (PDF) [1]. This equation is used in various fields such as aerospace, physics,

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chemistry, biology and finance. From the stochastic differential equation (SDE), the Fokker–Planck equation is obtained through the Kramers–Moyal expansion [2].

The exact solution of nonlinear stochastic differential systems is difficult in general cases, and the Fokker–Planck equation is an important tool. Many authors analyze linear and nonlinear SDEs using analytic and numerical approaches. For nonlinear systems, the equivalent linearization procedure is mostly used [3]: it is equivalent to exponential closure procedures in the case of Gaussian white noise excitation [4]. In the case of non-Gaussian white noise, another approach to approximate the partial differential equation of systems is the non-Gaussian closure method [5]. There are many numerical methods to solve the Fokker–Planck equation. Hesam et. al [6] elaborated an analytical solution based on the differential transform method. He's variational iteration method was implemented to solve the Fokker–Planck equation in [7]. The authors of [8] applied the combined Hermite spectral and upwinding difference methods and showed that the Hermite-based spectral methods were convergent with spectral accuracy in a weighted Sobolev space. Ben Said et. al [9] investigated random differential equations with uncertain parameters based on the polynomial chaos. In [10], the finite element and finite difference methods were used to solve the transient Fokker–Planck equation. The stationary Fokker–Planck equation was solved in [11] using the finite element method; the method was also compared with the Galerkin method. In [12], finite difference and element finite methods are applied to Van der Pol and Duffing oscillators for higher-dimensional systems. Global weighted residual and extended orthogonal functions methods are presented in [13, 14] to solve the Fokker–Planck equation associated to nonlinear stochastic systems.

This paper is an extension of the work originally presented at the World Conference on Complex Systems [15] and is focused on the elaboration of numerical procedures for nonlinear SDEs with uncertain parameters. Methodological approaches based on exponential closure combined with Monte Carlo and conditional expectation methods, as well as the meshfree method with radial basis functions combined with Monte Carlo and conditional expectation methods, are elaborated.

2. Problem Formulation

Define the stochastic differential system by:

$$\frac{dX_i}{dt} = g_i(X, \omega) + h_{ij}(X, \omega) W_j, \quad 1 \le i, j \le n, \tag{1}$$

where X_i is the i^{th} component of the response vector X, $W_j(t)$ is a stochastic excitation, and the functions g_i, h_{ij} are polynomial functions on X with random coefficients depending on the random vector ω.

This random vector has a distribution function with respect to the Lebesgue measure denoted by f. This vector is explicitly given by $\omega = (\xi_1, \ldots, \xi_n)$, where ξ_i for $i = 1$ to n are random variables defined from the probabilistic field $(\Omega_i, \mathbb{F}_i, \mathbb{P}_i)$ to $\mathbb{R}$. The excitation $W_j(t)$ is assumed to be a Gaussian white-noise process with the following properties:

$$E\left(W_j(t)\right) = 0 \tag{2}$$

$$E\left(W_j(t) W_k(t+\tau)\right) = \sigma_{jk}\delta(\tau), \tag{3}$$

where $\delta(\tau)$ is the Dirac function and σ_{jk} is the frequential density of the process W_j, W_k. The system response X for each given observation ω is a Markov process. Its conditional distribution function relative to an observation is defined by the following FPK equation:

$$\frac{\partial P}{\partial t} + \frac{\partial}{\partial x_i}(U_i P) - \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}\left(V_{ij}P\right) = 0 \tag{4}$$

$$P(x, t / x_0, t_0, w) = \prod_{j=1}^{n} \delta\left(x_j - x_{0j}\right), \tag{5}$$

where $P = P(x, t / x_0, t_0, \omega)$, x_j are the state variables of the first system, the functions U_i, V_{ij} are given respectively from the stochastic system equation (1) by $U_i(x, w) = g_i(x, w)$ and

$$V_{ij}(x) = 2\sigma_{ls} h_{il}(x, w) h_{js}(x, w).$$

$x = (x_1, \ldots, x_n)$ is the state vector associated to the stochastic system equation (1). In the stationary case, the conditional distribution function P is time independent and the reduced FPK equation, in this case, is given by:

$$\frac{\partial}{\partial x_i}(U_i P) - \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}\left(V_{ij}P\right) = 0, \tag{6}$$

with the following boundary condition:

$$\lim_{|x| \to +\infty} P(x / x_0, t_0, \omega) = 0 \tag{7}$$

$$\int_{\mathbb{R}^n} P(x / x_0, t_0, \omega)\, dx = 1. \tag{8}$$

The aim of this paper is the elaboration of efficient numerical methodological approaches to solve this last problem.

3. Methodological Approaches

3.1 Exponential Closure Combined with Monte Carlo Method

Let a sample of the random vector ω be constituted by an observation denoted $\{\omega^n, \text{ for } n = 1 \text{ to } N\}$. For each observation, let the function $Q(x, \omega^n)$ be defined by $Q(x, \omega^n) = \ln P(x, \omega^n)$; these functions $Q(x, \omega^n)$ and $P(x, \omega^n)$ are defined by Q^n and P^n.

These functions satisfy the following deterministic function deduced from the FPK equation (6) by:

$$\frac{\partial}{\partial x_j}(U_j) + U_j \frac{\partial}{\partial x_j}(Q^n) - \frac{1}{2}\left(\frac{\partial^2}{\partial x_i \partial x_j}(V_{ij}) + \frac{\partial}{\partial x_i}(V_{ij})\frac{\partial}{\partial x_j}(Q^n) + \frac{\partial}{\partial x_j}(V_{ij})\frac{\partial}{\partial x_i}(Q^n) + V_{ij}\frac{\partial^2}{\partial x_i \partial x_j}(Q^n) + V_{ij}\frac{\partial}{\partial x_i}(Q^n)\frac{\partial}{\partial x_j}(Q^n)\right) = 0. \tag{9}$$

The solution of this equation gives the logarithm of the conditional distribution function P^n. Unfortunately, the solution of this equation is in general not explicit. For this reason, approximations of Q^n are elaborated by many authors for SDEs with deterministic coefficients. This approximation can be given by a polynomial function on X. The coefficient of this polynomial function can be described by condensed vector a where $a^n \in \mathbb{R}^{\mathbb{M}}$, and the approximated solution $\tilde{P}^n$ of this equation is given by:

$$\begin{cases} \tilde{P}^n_{n_1}(X; a^n) = Ce^{\tilde{Q}^n_{n_1}(X; a^n)} & (X; a^n) \in D \times \mathbb{R}^M \\ \tilde{P}^n_{n_1}(X; a^n) = 0 & \text{otherwise,} \end{cases} \tag{10}$$

where C is a normalization constant and D is a bounded domain. The polynomial function $\tilde{Q}^n{}_{n_1}(X; a^n)$ is given by:

$$\tilde{Q}^n_{n_1}(X; a^n) = \sum_{i_1+\cdots+i_q=0}^{n_1} a^n_{i_1\ldots,i_q} x_1^{i_1} \ldots x_q^{i_q}. \tag{11}$$

It has to be noted that this polynomial function is a polynomial representation on x_j. Various polynomial orders can be considered, leading to different approximation orders. This expression is inserted in equation (9), and the residual error of this approximation is given by:

$$\begin{aligned} r_{n_1}^n(x; a^n) = \\ &\frac{\partial}{\partial x_j}\left(V_j\right) + V_j \frac{\partial}{\partial x_j}(\tilde{Q}_{n_1}^n) - \frac{1}{2}\left(\frac{\partial^2}{\partial x_i \partial x_j}\left(U_{ij}\right) + \frac{\partial}{\partial x_i}\left(U_{ij}\right)\right. \\ &\qquad \frac{\partial}{\partial x_j}(\tilde{Q}_{n_1}^n) + \frac{\partial}{\partial x_j}\left(U_{ij}\right)\frac{\partial}{\partial x_i}(\tilde{Q}_{n_1}^n) + \\ &\qquad \left.\frac{\partial^2}{\partial x_i \partial x_j}(\tilde{Q}_{n_1}^n) + U_{ij}\frac{\partial}{\partial x_i}(\tilde{Q}_{n_1}^n)\frac{\partial}{\partial x_j}(\tilde{Q}_{n_1}^n)\right). \end{aligned} \tag{12}$$

To determine the vector parameters, a variational procedure is used. The residual error $r_{n_1}^n$ is then projected on the Hilbert space generated by the given function:

$$\psi_{j_1 \ldots, j_q} = x_1^{j_1} \ldots x_q^{j_q} \varphi(x_1, \ldots, x_q), \tag{13}$$

where $1 \leq j_1 + \cdots + j_q \leq n_1$ and $\varphi\left(x_1, \ldots, x_q\right)$ is a Gaussian distribution that will be given later. The resulting expression is integrated with respect to $x_1, \ldots, x_q$, and the following functional F results for $1 \leq k \leq N$:

$$F_{n_1}^n(a^n) = \int_{\mathbb{R}^q} r_{n_1}^n(X; a^n) x_1^{j_1} \ldots x_q^{j_q} \varphi\left(x_1, \ldots, x_q\right) dx_1 \ldots dx_q. \tag{14}$$

The vector parameters a^n are obtained numerically by solving the following nonlinear algebraic system:

$$F_{n_1}^n(a^n) = 0 \text{ for } 1 \leq n_1 \leq N. \tag{15}$$

3.2 Exponential Closure Combined with Conditional Expectation Method

This method is used by Azrar et. al [16] for random differential equations with uncertain parameters, and consists of the determination of the expectation of the distribution function associated to the state of the random SDE defined in equation (1).

Let $P(x / \omega)$ be the distribution function solution of the stationary Fokker–Planck equation defined in equation (6); this distribution function depends on the random vector ω. The expectation of the distribution function with respect to the random variable ω is given by

$$P(x) = \int_{\mathbb{R}^n} P(x / \omega) f(\omega) \mathrm{d}\omega. \tag{16}$$

Assume that $\omega \in \Omega$, where Ω is a bounded domain in $\mathbb{R}^q$. The resulting reduced integral is approximated using the Gauss–Legendre quadrature method with the following Gauss points and weights:

$$\left\{\overline{\omega_i},\ 0 \leq i \leq L\right\} \text{ and } \left\{\overline{p_i},\ 0 \leq i \leq L\right\}. \tag{17}$$

The integral equation (16) is thus approximated by

$$P(x) \simeq \sum_{i=0}^{L} P(x / \omega_i) f(\omega_i) \overline{p_i}. \tag{18}$$

To get the values of $P(x / \omega_i)$, the exponential closure is used. This function is denoted by P^i and its logarithm is introduced and denoted by Q^i.

The same procedure developed in the last paragraph is adopted herein, and the function Q^i is the solution of the partial differential equation deduced from the Fokker–Planck equation associated to P^i, where the random vector ω is replaced by the observation ω_i.

The solution Q^i is then approximated by a polynomial function of the state x. The coefficient of this approximation is determined using the same projection on a Hilbert space used in the last subsection. The resulting algebraic equation is solved using the Newton–Raphson method, and the solution is thus approximated.

3.3 Exponential Closure Method

The exponential closure method is based on the hypothesis that the PDF of the responses of a nonlinear stochastic equation is assumed to be an exponential function of the polynomial in state variables. This method is used by many authors, when the diffusion and drift coefficients of the SDE equation (1) are deterministic [4]. In this paper, the exponential closure method is based on the random diffusion and drift of SDE. More details about this method are presented in [15].

3.4 Meshfree Method: Radial Basis Functions

RBFs based on collocation methods were first used by Kansa in 1991 [17]; it is considered a meshfree method [18] and used to solve partial differential equations. There are many RBFs defined by a shape parameter; this shape parameter is important for convergence of the RBF method. This shape parameter is chosen arbitrarily, but many authors used optimization approaches to optimize it [19, 20].

Let N be arbitrary points $X_1, X_2, \ldots, X_N$ in an open set Ω of $\mathbb{R}^d$. We consider n points inside Ω and $m = N - n$ on boundary $\partial\Omega = \Gamma$ $(x_j)_{(1 \leq j \leq n)} \subset \Omega$ and $(x_j)_{(n+1 \leq j \leq m)} \subset \Gamma$.

To illustrate this method, we consider the stationary Fokker–Planck equation (6) and the boundary conditions (7) and (8). For more details on this method, see [15].

3.5 Radial Basis Function Combined with Monte Carlo Method

Let a sample of the random vector ω be constituted by an observation denoted by $\{\omega^n, \text{ for } n = \text{to } N_1\}$. For each observation, let the function $P(x, \omega^n)$ be denoted by P^n.

To approximate this function, an RBF method is used. For this aim, let us consider N arbitrary points $X_1, X_2, \ldots, X_N$ in an open set Ω of $\mathbb{R}^d$. We consider n points inside Ω and $m = N - n$ on boundary $\partial\Omega = \Gamma\ (x_j)_{(1 \le j \le n)} \subset \Omega$ and $(x_j)_{(n+1 \le j \le m)} \subset \Gamma$.

P^n verifies the following stationary Fokker–Planck equation and the boundary conditions:

$$\frac{\partial}{\partial x_i}(U_i P^n) - \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}\left(V_{ij} P^n\right) = 0$$

$$\lim_{|x| \to +\infty} P^n(x / x_0) = 0$$

$$\int_{\mathbb{R}^n} P^n(x / x_0) dx = 1.$$

These equations are rewritten in the following usual form:

$$LP^n(x) = f(x) \quad x \in \Omega \subset \mathbb{R}^d \tag{19}$$

$$BP^n(x) = T(x), \quad x \in \Gamma, \tag{20}$$

where L, B are linear operators; in this case the B specifies a Dirichlet boundary condition.

Let $\tilde{P}^n$ be an approximate solution of P^n in the following form:

$$\tilde{P}^n(x) = \sum_{j=1}^{N} \alpha_j \phi_j(x), \tag{21}$$

where $\phi_j(x) = \phi\left(\| x - x_j \|_2\right)$, ϕ is an RBF and α_j are unknowns.

In this paper, the multiquadric RBF $\phi_j(r) = \sqrt{r^2 + c^2}$ where $c \ge 0$ is the shape parameter used.

Substituting equation (21) into equations (19) and (20) leads to:

$$L\tilde{P}^n = \sum_{j=1}^{N} \alpha_j L\phi_j(x_i) = f(x_i) \quad 1 \le i \le n,$$

$$B\tilde{P}^n = \sum_{j=1}^{N} \alpha_j B\phi_j(x_i) = T(x_i) \quad n + 1 \le i \le N.$$

Then, we get a linear system of unknowns $\alpha_{j=1}^{n+m}$ $(n+m=N)$ to solve:

$$\begin{pmatrix} L_{1,1}P^n & \ldots & L_{1,N}P^n \\ & \ldots & \\ & \ldots & \\ & \ldots & \\ & \ldots & \\ B_{N,1}P^n & \ldots & B_{n+m,N}P^n \end{pmatrix}\begin{pmatrix} \alpha_1 \\ . \\ . \\ . \\ . \\ \alpha_N \end{pmatrix} = \begin{pmatrix} f_1 \\ . \\ . \\ . \\ . \\ T_N \end{pmatrix}.$$

Based on the solution of this system, the approximate solution $\tilde{P}^n$ is calculated by equation (21) and the mean of the solution is given by the following equation:

$$\overline{P^n} = \frac{1}{N}\sum_{i=1}^{N} P^n(X_i). \tag{22}$$

3.6 Radial Basis Function Combined with Conditional Expectation Method

In this subsection, we consider the Gauss and weight points defined in equation (17), and for each Gauss point ω_i, the conditional distribution function associated to this observation $P(x\,/\,\omega_i)$ is denoted by P^i. This function is a solution of the deterministic Fokker–Planck equation associated to the original SDE equation (1) for a given observation ω_i. The RBF method developed previously is used herein, and an approximate solution is then obtained. To give the expectation of the associated distribution function with respect to the random variable ω, a Gauss–Legendre quadrature method is used, and the same equation defined by equation (18) is obtained.

4. Numerical Results and Discussion

In order to highlight the effectiveness of the methodological approaches presented, some nonlinear random differential equations with random parameters are investigated.

4.1 Application 1

Let us consider the nonlinear diffusion process $X(t)$ defined by the following SDE:

$$\dot{X} = \frac{1}{2}\left(X - X^3 - \varepsilon X^5\right) + W(t), \tag{23}$$

where $W(t)$ is a Gaussian white noise with the autocorrelation function given by $E(W(t)W(t+\tau)) = 2\sigma_0\delta(\tau)$, and σ_0 is the spectral density of $W(t)$. ε is a random variable with mean ε_0 and standard deviation ε_1 for $\sigma_0 = 1$. The exact conditional distribution function, given an observation of the random variable ε, is given by:

$$P\left(\frac{x}{\varepsilon}\right) = C\exp\left(\frac{x^2\left(1-\left(x^2/2\right)-\varepsilon\left(x^4/2\right)\right)}{4}\right), \tag{24}$$

where C is a normalized constant.

Let us define the centered and normalized random variable ε by $\xi = (\varepsilon - \varepsilon_0)/\varepsilon_1$. In this case, the stationary Fokker–Planck equation associated to equation (23) is the following:

$$\frac{\partial}{\partial x}(gP) - \frac{1}{2}\frac{\partial^2}{\partial x^2}(hP) = 0,$$

where $g(x) = 1/2\left(x - x^3 - \varepsilon x^5\right)$ and $h(x) = \sigma^2/2$ with $\sigma = \sqrt{2}$, and the shape parameter of the RBF method is $c = 0.1$.

The presented methods are applied to the stationary Fokker–Plank equation and the following results are obtained.

Figure 1 shows the logarithmic conditional distribution function for degree 4 of exponential closure combined with the conditional expectation method in the stationary case. It is observed that the approximate logarithmic distribution coincides perfectly with the exact solution.

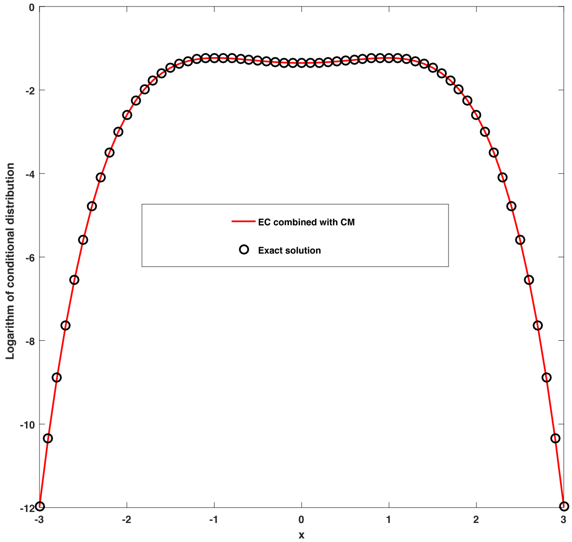

Figure 1. Logarithmic conditional distribution function of the state variable x in the stationary case obtained by exponential closure degree 4 combined with the conditional expectation method.

Figure 2 shows the logarithmic conditional distribution function for degree 4 of exponential closure combined with Monte Carlo method (1000) random numbers in the stationary case. Small discrepancies are observed with the exact solution and, particularly, far from the center.

Figure 3 shows the logarithmic conditional distribution function obtained by the meshfree method with the multiquadric RBF combined with the conditional expectation method. The shape parameter used in this case is $c = 0.1$. It is observed that the approximate logarithmic distribution coincides perfectly with the exact solution.

Figure 4 gives the logarithmic conditional distribution function obtained by the meshfree method with the multiquadric RBF combined with Monte Carlo method (1000) random numbers in the stationary case. The shape parameter used is $c = 0.1$. It is clearly observed that the approximate logarithmic distribution coincides perfectly with the exact solution.

Figure 5 gives the logarithmic conditional distribution function for all methods used, where the shape parameter is $c = 0.1$ for RBF. It is noticed that these approximate solutions are in good agreement with each other and with the exact solution.

Figure 6 gives the distribution function by all presented methods where the shape parameter is $c = 0.1$ for the RBF. It is noticed that these distribution functions coincide between them and coincide perfectly with the exact solution.

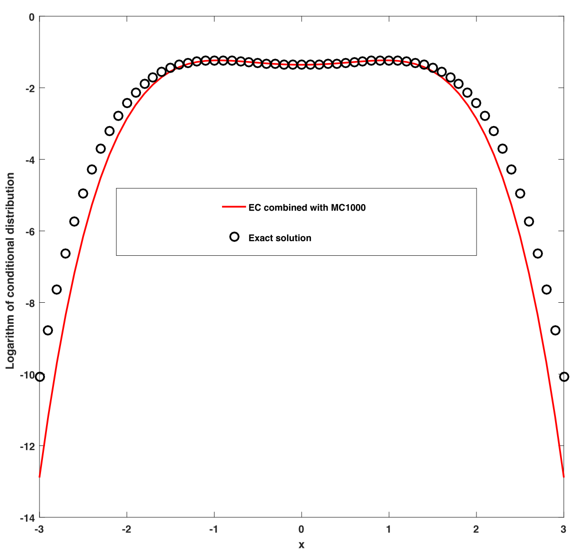

Figure 2. Logarithmic conditional distribution function of the state variable x in the stationary case obtained by exponential closure degree 4 combined with the Monte Carlo (1000) method.

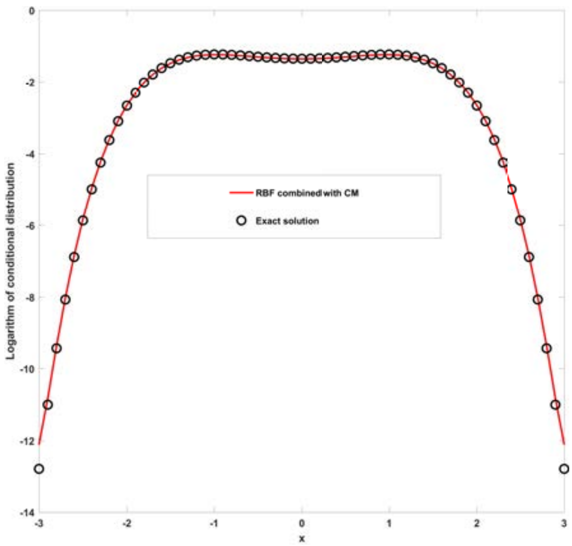

Figure 3. Logarithmic conditional distribution function of the state variable x in the stationary case obtained by the RBF combined with the conditional method.

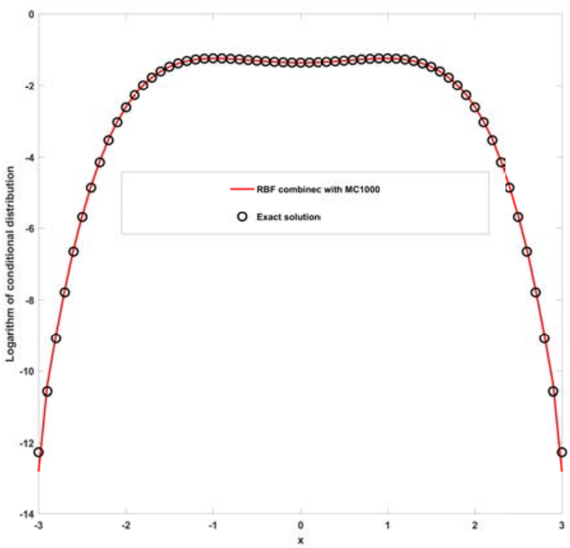

Figure 4. Logarithmic conditional distribution function of the state variable x in the stationary case obtained by the RBF combined with the Monte Carlo (1000) method.

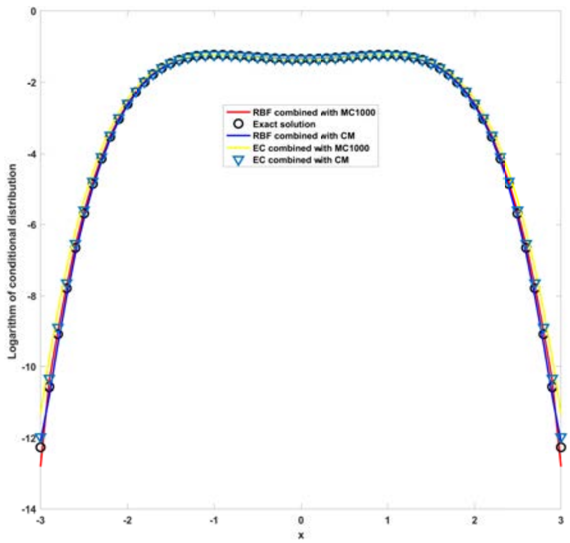

Figure 5. Logarithmic conditional distribution function of the state variable x in the stationary case of the proposed methods.

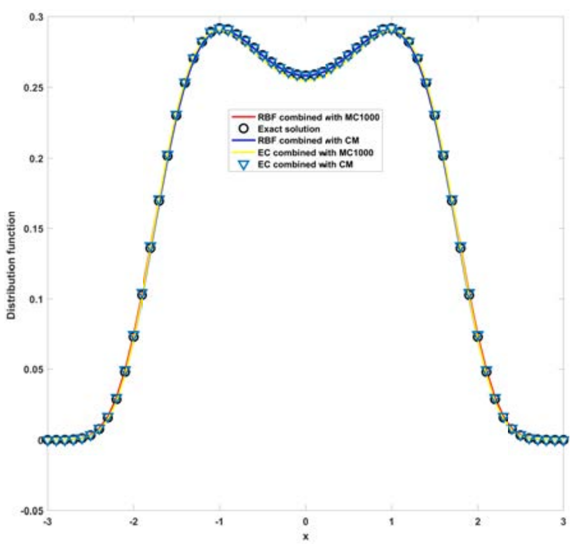

Figure 6. Distribution function of the state variable x in the stationary case of the proposed methods.

4.2 Application 2

Let us consider the nonlinear dynamic behavior of an Euler–Bernoulli beam studied by G. K. Er in [21] governed by the following nonlinear partial differential equation:

$$\begin{gathered}\rho A\ddot{Y}(X,t)+C\dot{Y}(X,t)+EI\frac{\partial^4 Y(X,t)}{\partial X^4}-\\ \frac{EA}{2I}\frac{\partial^2 Y(X,t)}{\partial X^2}\int_0^L\left(\frac{\partial Y\left(X',t\right)}{\partial X'}\right)^2 dX' = qw(t)\end{gathered} \tag{25}$$

with the following boundary condition:

$$Y\left(0,t\right)=Y(L,t)=\frac{\partial^2 Y\left(0,t\right)}{\partial X^2}=\frac{\partial^2 Y(L,t)}{\partial X^2}=0, \tag{26}$$

where ρ, A, E and I are random variables and $W(t)$ is a Gaussian process with zero mean, with the correlation function defined by:

$$E(W(t)W(t+\tau))=2\sigma\delta(\tau). \tag{27}$$

Using the Galerkin projection method, the solution $Y(X,t)$ is approximated by:

$$Y(X,t)=\sum_{i=1,3}^{2m-1}Z_i(t)\sin\left(\frac{(i\pi X)}{L}\right). \tag{28}$$

For the sake of simplicity, the first one mode, $m=1$, is considered. The modal analysis leading to a coupled nonlinear differential system can also be investigated with one mode. The following stochastic nonlinear differential equation with nonlinear random parameters results:

$$\ddot{Z}_1+\frac{C}{\rho A}\dot{Z}_1+\frac{EI\pi^4}{\rho A L^4}Z_1+\frac{E\pi^4}{4\rho L^4}Z_1^3=\frac{4q}{\pi\rho A}W(t). \tag{29}$$

To describe the FPK equation associated to this equation for the conditional distribution with respect to the random variables ρ, A, E and I, equation (29) is reformulated as:

$$\begin{cases}\dot{Z}_1 &= Z_2\\ \dot{Z}_2 &= -\dfrac{C}{\rho A}Z_2-\dfrac{EI\pi^4}{\rho A L^4}Z_1-\dfrac{E\pi^4}{4\rho L^4}Z_1^3+\\ &\quad \dfrac{4q}{\pi\rho A}W(t) \quad \text{otherwise.}\end{cases} \tag{30}$$

The functions h and g related to the system (1) are in this case defined by:

$$g(Z_1, Z_2, w) = \left(Z_2, -\frac{C}{\rho A}Z_2 - \frac{EI\pi^4}{\rho A L^4}Z_1 - \frac{E\pi^4}{4\rho L^4}Z_1^3\right) \tag{31}$$

and

$$h(Z_1, Z_2, w) = \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{4q}{\pi\rho A} \end{bmatrix}. \tag{32}$$

The functions U and V are given by:

$$U_1(Z_1, Z_2, \omega) = Z_2 \tag{33}$$

$$U_2(Z_1, Z_2, \omega) = -\frac{C}{\rho A}Z_2 - \frac{EI\pi^4}{\rho A L^4}Z_1 - \frac{E\pi^4}{4\rho L^4}Z_1^3 \tag{34}$$

$$V_{22}(Z_1, Z_2, \omega) = 2\sigma\left(\frac{4q}{\pi\rho A}\right)^2. \tag{35}$$

The other components of V are null. The FPK equation associated to this system in the stationary case is given by:

$$\begin{aligned} &Z_2\frac{\partial}{\partial Z_1}(P) - \left(\frac{C}{\rho A}Z_2 + \frac{EI\pi^4}{\rho A L^4}Z_1 + \frac{E\pi^4}{4\rho L^4}Z_1^3\right)\frac{\partial}{\partial Z_2}(P) - \\ &\quad \frac{C}{\rho A}P - \sigma\left(\frac{4q}{\pi\rho A}\right)^2 \frac{\partial^2}{\partial Z_2^2}(P) = 0. \end{aligned} \tag{36}$$

The solution of this equation gives the conditional distribution P in the stationary case for equation (29), and the exact distribution solution is given by:

$$\begin{aligned} &P\left(Z_1, \frac{Z_2}{\rho}, A, E, I\right) = \\ &\quad C\exp\left(-\frac{C\pi^2\rho A}{2\sigma(4q)^2}\left(Z_2^2 + \frac{EI\pi^4}{\rho A L^4}Z_1^2 + \frac{E\pi^4}{8\rho L^4}Z_1^4\right)\right). \end{aligned} \tag{37}$$

The joint distribution function of the state vector (Z_1, Z_2) is given by

$$\begin{aligned} &P(Z_1, Z_2) = \\ &\quad \int_\Omega C\exp\left(-\frac{C\pi^2\rho A}{2\sigma(4q)^2}\left(Z_2^2 + \frac{EI\pi^4}{\rho A L^4}Z_1^2 + \frac{E\pi^4}{8\rho L^4}Z_1^4\right)\right) \\ &\qquad f(\rho, A, E, I)dw, \end{aligned} \tag{38}$$

where the parameters ρ, C, L, E and I are random variables.

We apply the proposed methods for the stationary equation (36), and the obtained numerical results compare well with the exact solution.

Figure 7 indicates the logarithmic conditional distribution function of the beam of exponential closure for degree 4 combined with the conditional method compared with the exact solution.

Figure 8 gives the logarithmic conditional distribution function of the beam of exponential closure for degree 4 combined with the Monte Carlo ($N = 1000$) method compared with the exact solution.

Figure 9 shows the logarithmic conditional distribution function of the beam of the meshfree method with multiquadric RBF combined with the conditional method compared with the exact solution.

Figure 10 gives the logarithmic conditional distribution function of the beam of multiquadric RBF combined with the Monte Carlo method for 1000 random variables compared with the exact solution.

Figure 11 indicates the comparison between the proposed methods of the beam; it is noticed that the distribution functions coincide perfectly with the exact solution.

Figure 12 indicates the comparison between the proposed methods of the beam; it is noticed that the logarithmic conditional distribution functions coincide perfectly with the exact solution.

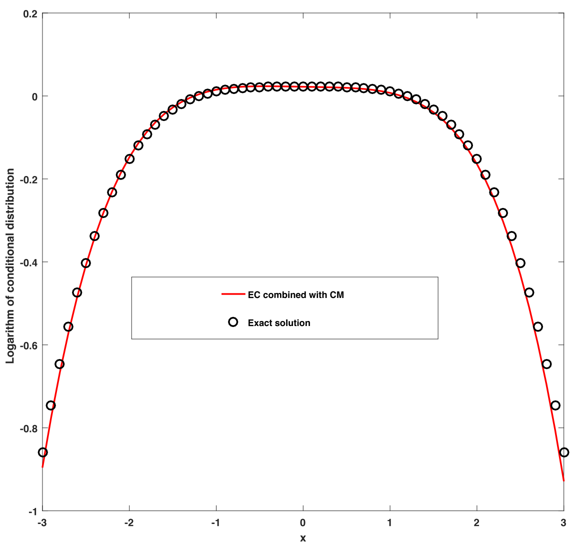

Figure 7. Logarithmic conditional distribution function of the state variable x in the stationary case obtained by exponential closure degree 4 combined with the conditional method.

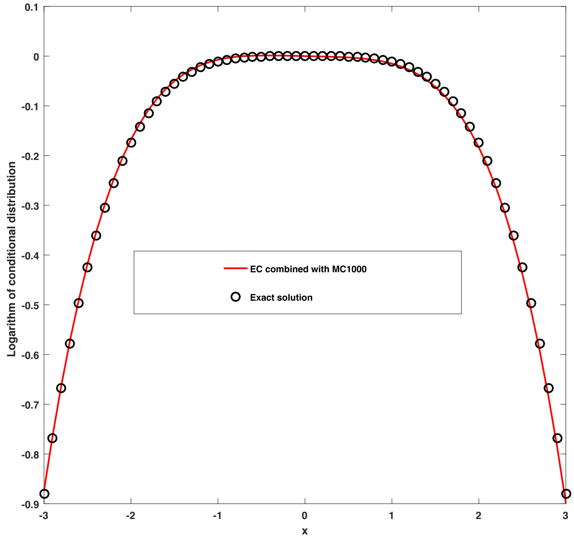

Figure 8. Logarithmic conditional distribution function of the state variable x in the stationary case obtained by exponential closure degree 4 combined with the Monte Carlo 1000 method.

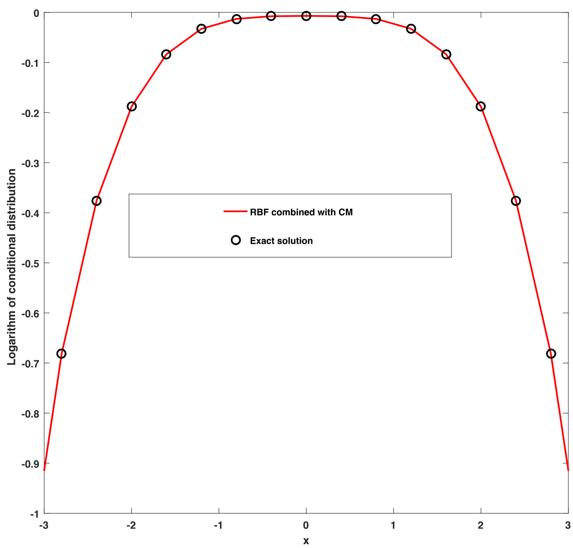

Figure 9. Logarithmic conditional distribution function of the state variable x in the stationary case obtained by the RBF combined with the conditional method.

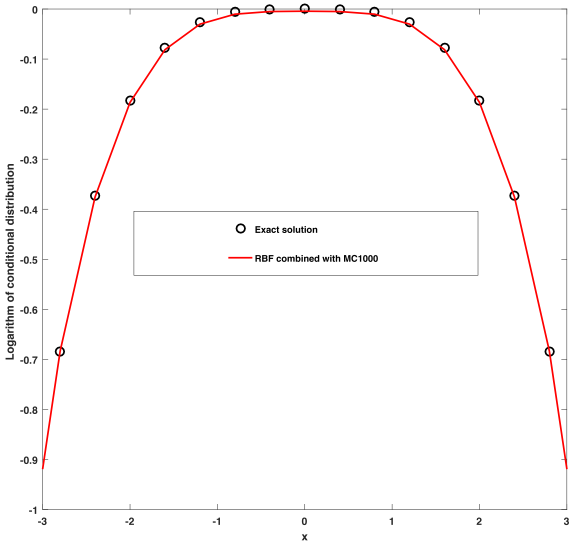

Figure 10. Logarithmic conditional distribution function of the state variable x in the stationary case obtained by the RBF combined with the Monte Carlo 1000 method.

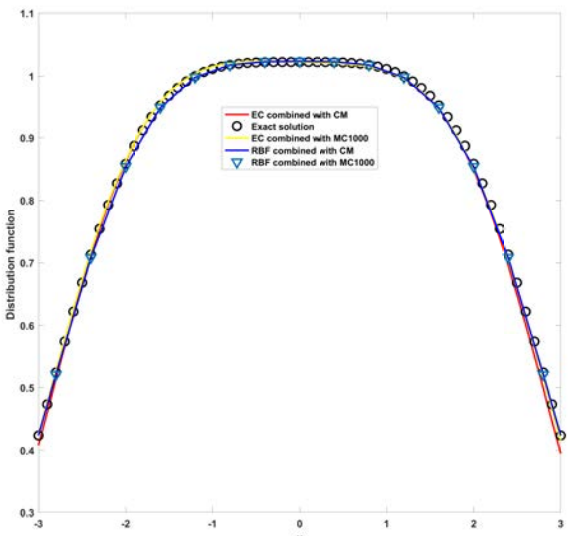

Figure 11. Distribution function of the state variable x in the stationary case of the proposed methods.

https://doi.org/10.25088/ComplexSystems.28.4.411

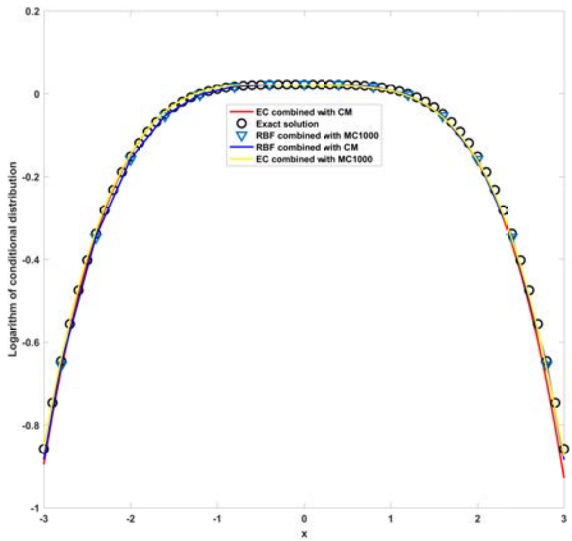

Figure 12. Logarithmic conditional distribution function of the state variable x in the stationary case of the proposed methods.

5. Conclusion

This work is based on numerical approaches: the meshfree method with a multiquadric radial basis function (RBF) combined with the expectation conditional method and with the Monte Carlo approach on one hand, and exponential closure combined with the conditional method and the Monte Carlo method on the other hand, are analyzed for a nonlinear stochastic differential equation (SDE) with random parameters. The stationary Fokker–Planck equation for nonlinear SDE and the Euler–Bernoulli beam with hinged end supports are calculated when the uniformly distributed lateral force is a Gaussian white noise. The numerical solutions given by the proposed methods coincide perfectly with the exact solution. The results showed that these proposed approaches are efficient and can be extended to more general SDEs with random parameters.

This work elaborated coupling methodological approaches for nonlinear stochastic differential equations with uncertain parameters. These methods are on one hand the meshfree method with a multiquadric radial basis function (RBF) combined with expectation conditional method and with the Monte Carlo approach, and exponential closure combined with the conditional method and the Monte Carlo approach on the other hand. The stationary Fokker–Planck equation for a nonlinear stochastic differential equation (SDE) and for the

Euler–Bernoulli beam with hinged end supports and uncertain parameters are numerically analyzed when the uniformly distributed lateral force is a Gaussian white noise. The numerical solutions given by the proposed coupling methods coincide perfectly with the exact solution. The obtained results showed that these proposed approaches are efficient and can be extended to more general SDEs with random parameters.

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References

[1] W. Martens and U. von Wagner, "Calculation of Probability Density Functions for Nonlinear Vibration Systems," *Proceedings in Applied Mathematics and Mechanics*, **11**(1), 2011 pp. 923–926. doi:10.1002/pamm.201110440.

[2] H. Risken, "Fokker–Planck Equation," *The Fokker–Planck Equation*, Berlin, Heidelberg: Springer, 1996 pp. 63–95. doi:10.1007/978-3-642-61544-3_4.

[3] A. Brückner and Y. K. Lin. "Generalization of the Equivalent Linearization Method for Non-Linear Random Vibration Problems," *International Journal of Non-Linear Mechanics*, **22**(3), 1987 pp. 227–235. doi:10.1016/0020-7462(87)90005-9.

[4] G.-K. Er, "Exponential Closure Method for Some Randomly Excited Non-linear Systems," *International Journal of Non-Linear Mechanics*, **35**(1), 2000 pp. 69–78. doi:10.1016/S0020-7462(98)00088-2.

[5] S. H. Crandall, "Non-Gaussian Closure Techniques for Stationary Random Vibration," *International Journal of Non-Linear Mechanics*, **20**(1), 1985 pp. 1–8. doi:/10.1016/0020-7462(85)90042-3.

[6] S. Hesam, A. R. Nazemi and A. Haghbin, "Analytical Solution for the Fokker–Planck Equation by Differential Transform Method," *Scientia Iranica*, **19**(4), 2012 pp. 1140–1145. doi:10.1016/j.scient.2012.06.018.

[7] M. Dehghan and M. Tatari, "The Use of He's Variational Iteration Method for Solving a Fokker–Planck Equation," *Physica Scripta*, **74**(3), 2006 pp. 310–316. doi:10.1088/0031-8949/74/3/003.

[8] J. C. M. Fok, B. Guo and T. Tang, "Combined Hermite Spectral-Finite Difference Method for the Fokker–Planck Equation," *Mathematics of Computation*, **71**(240), 2001 pp. 1497–1528. doi:10.1090/S0025-5718-01-01365-5.

[9] M. B. Said, L. Azrar and D. Sarsri, "Numerical Procedures for Random Differential Equations," *Journal of Applied Mathematics*, 2018 7403745. doi:10.1155/2018/7403745.

[10] L. Pichler, A. Masud and L. A. Bergman, "Numerical Solution of the Fokker–Planck Equation by Finite Difference and Finite Element Methods—A Comparative Study," *Computational Methods in Stochastic Dynamics* (M. Papadrakakis, G. Stefanou and V. Papadopoulos, eds.), Dordrecht: Springer, 2013 pp. 69–85. doi:10.1007/978-94-007-5134-7_5.

[11] R. S. Langley, "A Finite Element Method for the Statistics of Non-linear Random Vibration," *Journal of Sound and Vibration*, **101**(1), 1985 pp. 41–54. doi:10.1016/S0022-460X(85)80037-7.

[12] P. Kumar and S. Narayanan, "Solution of Fokker–Planck Equation by Finite Element and Finite Difference Methods for Nonlinear Systems," *Sadhana*, **31**(4), 2006 pp. 445–461. doi:10.1007/BF02716786.

[13] U. von Wagner and W. V .Wedig, "On the Calculation of Stationary Solutions of Multi-Dimensional Fokker–Planck Equations by Orthogonal Functions," *Nonlinear Dynamics*, **21**(3), 2000 pp. 289–306. doi:10.1023/A:1008389909132.

[14] W. V. Wedig and U. Von Wagner, "Extended Laguerre Polynomials for Nonlinear Stochastic Systems," *Computational Stochastic Mechanics* (P. D. Spanos, ed.), Brookfield, VT: A. A. Balkema Publishers, 1999 pp. 293–298.

[15] I. Salleh, M. Ben Said and L. Azrar, "Numerical Simulation of Nonlinear Stochastic Differential System with Uncertain Parameters," in *4th World Conference on Complex Systems (WCCS)*, Ouarzazate, Morocco, IEEE, forthcoming.

[16] A. Azrar, M. B. Said, L. Azrar and A. A. Aljinaidi, "Dynamic Analysis of Carbon Nanotubes Conveying Fluid with Uncertain Parameters and Random Excitation," *Mechanics of Advanced Materials and Structures*, **26**(10), 2019 pp. 898–913. doi:10.1080/15376494.2018.1430272.

[17] S. A. Sarra and E. J. Kansa, "Multiquadric Radial Basis Function Approximation Methods for the Numerical Solution of Partial Differential Equations," *Advances in Computational Mechanics*, **2**(2), 2009.

[18] G. E. Fasshauer, *Meshfree Approximation Methods with MATLAB*, Hackensack, NJ: *World Scientific*, 2007.

[19] I. Salleh, Y. Belkourchia and L. Azrar, "Optimization of the Shape Parameter of RBF Based on the PSO Algorithm to Solve Nonlinear Stochastic Differential Equation," *2019 5th International Conference on Optimization and Applications (ICOA)* Kenitra, Morocco, 2019 (H. Hachimi, et al., eds.), Piscataway, NJ: IEEE, 2019 pp. 203–207. doi:10.1109/ICOA.2019.8727661.

[20] J. A. Koupaei, M. Firouznia and S. M. M. Hosseini, "Finding a Good Shape Parameter of RBF to Solve PDEs Based on the Particle Swarm Optimization Algorithm," *Alexandria Engineering Journal*, **57**(4), 2018 pp. 3641–3652. doi:10.1016/j.aej.2017.11.024.

[21] G. K. Er, "The Probabilistic Solutions of Some Nonlinear Stretched Beams Excited by Filtered White Noise," *Procedia IUTAM*, **6**, 2013 pp. 141–150. doi:10.1016/j.piutam.2013.01.016.