Max-Plus Generalization of Conway's Game of Life

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We propose a max-plus equation that includes Conway's Game of Life (GoL) as a special case. There are some special solutions to the equation that include and unify solutions to GoL. Moreover, the multivalue extension of GoL is derived from the equation, and the behavior of solutions is discussed.

Keywords: cellular automaton; Conway's Game of Life; max-plus equation

1. Introduction

Conway's Game of Life (GoL) is a binary cellular automaton (CA) and expresses a kind of population ecology [1, 2]. It is an evolution game using a two-dimensional orthogonal grid of cells where each cell has one of two states, alive or dead. The evolution rule for the discrete generation is defined as follows.

1. Birth: If there are just three live cells in the Moore neighborhood of a dead cell, the dead cell changes to a live cell at the next generation.
2. Survival: If there are two or three live cells in the Moore neighborhood of a live cell, it is alive at the next generation.
3. Death: Otherwise, the cell is dead at the next generation.

Let us assume values of two states, 1 for alive and 0 for dead. If u_{ij}^n denotes the value at the (i, j) cell of the generation n, the above evolution rule can be transcribed into the evolution equation,

$$u_{ij}^{n+1} = \begin{cases} 1 & \left(u_{ij}^n, s_{ij}^n\right) = (0, 3),\ (1, 2) \text{ or } (1, 3) \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where

$$s_{ij}^n = u_{i-1j-1}^n + u_{ij-1}^n + u_{i+1j-1}^n + u_{i-1j}^n + u_{i+1j}^n + u_{i-1j+1}^n + u_{ij+1}^n + u_{i+1j+1}^n.$$

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Specific solutions to GoL have been extensively searched and listed. There are various types of evolution of solutions and they are analogous to activities of life.

In this paper, we propose an extended model of GoL using the max-plus operation. The max-plus operation is based on max-plus algebra, which is a commutative semiring defined by addition "max" and multiplication "+" [3]. It is used for the description of discrete event systems and utilized to analyze the dynamics with max-plus linear system theory based on the Perron–Frobenius theory [4]. It is also obtained by ultradiscretizing the difference equations through the limiting procedure,

$$\lim_{\varepsilon \to +0} \varepsilon \log\left(e^{A/\varepsilon} + e^{B/\varepsilon}\right) = \max\left(A, B\right). \tag{2}$$

Tokihiro et. al. found that the binary CA called a "box and ball system" is obtained by ultradiscretizing the discrete soliton equation through the above limit and showed that the solutions to the box and ball system giving soliton interactions among groups of balls can be derived by the same limit of multi-soliton solutions to the discrete equation [5, 6].

In the above context, we can consider that max-plus expression proposes a novel viewpoint and mathematical tools to pure digital systems like CA. We propose a max-plus equation with a continuous dependent variable in which GoL is embedded as a special case. There exist real-valued exact solutions to the equation and they include and unify solutions to GoL. Contents of this paper are as follows. In Section 2, the max-plus equation extended from GoL is proposed. In Section 3, special solutions to the max-plus equation and their relations to solutions to GoL are shown. In Section 4, we show the multivalue CA obtained from the max-plus equation and discuss the behavior of solutions. In Section 5, we give concluding remarks.

2. Definition of MaxLife

Let us consider the following evolution equation using operators max, + and − :

$$u_{ij}^{n+1} = F\left(u_{ij}^{n}, s_{ij}^{n}\right), \tag{3}$$

where i and j are integer space coordinates, n is integer time, s_{ij}^{n} is the sum of eight u's in the Moore neighborhood,

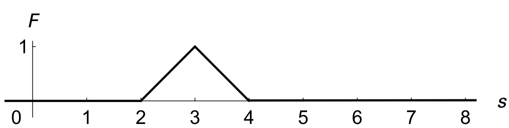
$$s_{ij}^{n} = u_{i-1j-1}^{n} + u_{ij-1}^{n} + u_{i+1j-1}^{n} + u_{i-1j}^{n} + u_{i+1j}^{n} + u_{i-1j+1}^{n} + u_{ij+1}^{n} + u_{i+1j+1}^{n},$$

and $F(u, s)$ is defined by

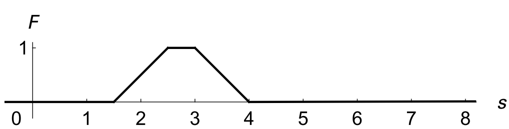
$$F(u, s) = \max\left(0, u + s - 2\right) - \max\left(0, u + s - 3\right) - \max\left(0, s - 3\right) + \max\left(0, s - 4\right).$$

If $0 \le u \le 1$, we can easily show $0 \le F(u, s) \le 1$. Figure 1 shows the graphs of $F\left(0, s\right)$, $F\left(0.5, s\right)$ and $F\left(1, s\right)$.

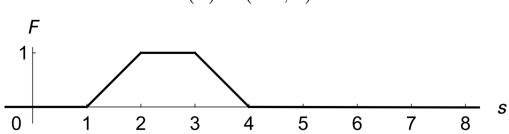
Consider the initial value problem for equation (3) and assume $n = 0$ is an initial time. If we set the initial data u_{ij}^{0} to satisfy $0 \le u_{ij}^{0} \le 1$ for any i and j, then any u_{ij}^{n} $(n > 0)$ also since $0 \le F(u, s) \le 1$. Moreover, if we assume u_{ij}^{n} at a certain n for any i and j takes either of the values 0 or 1, s_{ij}^{n} is one of the nine integer values from 0 to 8. Then the value of the right-hand side of equation (3) is also 0 or 1 considering the graphs of $F\left(0, s\right)$ and $F\left(1, s\right)$. Therefore, the value of solution u_{ij}^{n} to equation (3) can be closed in the binary set $\left\{0, 1\right\}$ if the initial data u_{ij}^{0} is. Then the evolution equation (3) becomes equivalent to equation (1) considering the profiles of $F\left(0, s\right)$ and $F\left(1, s\right)$. Thus equation (3) includes the rule of GoL as a special case. We call the evolution system defined by equation (2) "MaxLife" in this meaning and discuss the behavior of its real-valued solutions closed in the range of $\left[0, 1\right]$, relating them to binary solutions closed in the range of $\left\{0, 1\right\}$, which are also solutions to GoL.



(a) $F(0, s)$



(b) $F(0.5, s)$



(c) $F(1, s)$

Figure 1. Graphs of $F(u, s)$.

3. Special Solutions to MaxLife

In this section, we show the special solutions to MaxLife. Since it is difficult to solve equation (3) in a systematic way, we assume dimensions, symmetry and period of solutions within the background $u = 0$. The solutions shown in this section are confined at most to a 4×4 region and are static, periodic or moving stably. They reduce to binary solutions to GoL in a special case and most solutions unify two or more solutions to GoL.

3.1 Static Solution Confined in a 2 × 2 Region (Block)

The first example is a static solution confined in a 2×2 region,

$$\begin{array}{l} \ldots000000\ldots \\ \ldots00cd00\ldots \\ \ldots00ab00\ldots \\ \ldots000000\ldots \end{array},$$

where a, b, c and d are all real constants from 0 to 1. The region outside the one shown is $u = 0$. Substituting the static 2×2 solution into equation (3), we obtain

$$\begin{aligned} a = {} & \max(0, a+b+c+d-2) - \max(0, a+b+c+d-3) \\ & -\max(0, b+c+d-3) + \max(0, b+c+d-4). \end{aligned}$$

Since $a, b, c, d \in [0, 1]$, the equation reduces to

$$a = \max(0, a+b+c+d-2) - \max(0, a+b+c+d-3).$$

The right-hand side is symmetric about constants and the following condition is obtained considering other equations:

$$a = b = c = d, \quad a = \max(0, 4a-2) - \max(0, 4a-3).$$

Solving this condition, we have

$$a = b = c = d = 0 \text{ or } \frac{2}{3} \text{ or } 1.$$

These values give a trivial solution ($a = 0$), a noninteger solution ($a = 2/3$) and a binary solution to GoL called a "block" ($a = 1$).

3.2 Blinker Type of Solution

Solutions from this subsection are shown schematically as figures without proof. The four colored cells shown in Figure 2 are used to denote the values of u where a is any constant from 0 to 1. The "blinker" type of solution is shown in Figure 3. This solution is periodic with period 2. The double arrow "↔" means that at the next time step a left state will change to right or a right state will change to left. Since the blinker of GoL is obtained in the case of $a = 0$ and 1, the solution in Figure 3 includes two configurations of blinkers rotated 90 degrees to each other as shown in Figure 4.

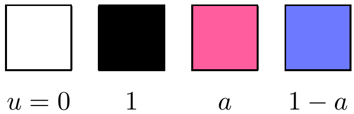


Figure 2. Four colored cells denoting the values of u.

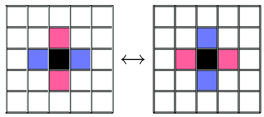


Figure 3. Blinker type of solution.

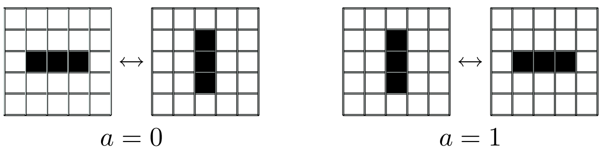


Figure 4. Blinker of GoL.

3.3 Clock Type of Solution

The examples shown in Figure 5 are five types of solutions giving the "clock" and another of GoL in a special case. All solutions give a clock for $a = 0$. For $a = 1$, a static solution of GoL or another clock rotated by 90 degrees is given. The solutions of GoL described here are shown in Figure 6. Note that the clock is periodic with period 2 and the other solutions are all static.

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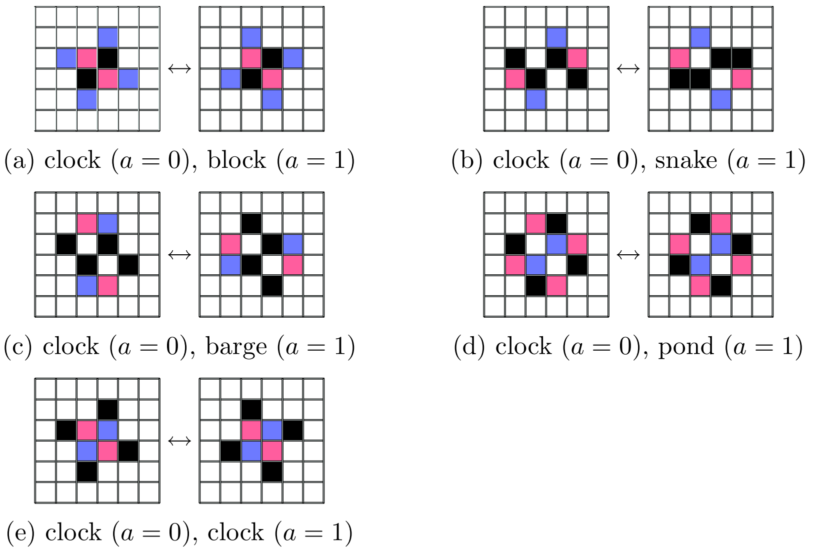


Figure 5. Solutions to equation (3) with period 2 giving a clock for $a = 0$. For $a = 1$, they give (a)–(d) a static solution or (e) a clock rotated by 90 degrees.

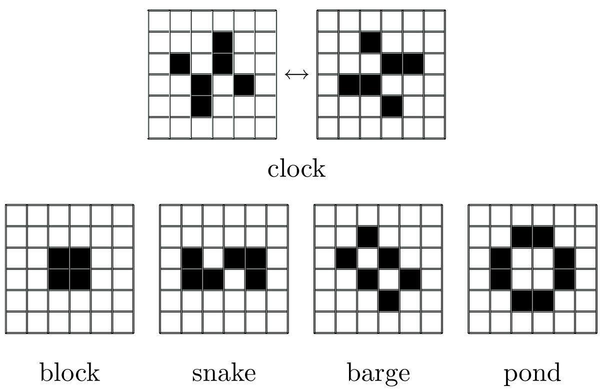
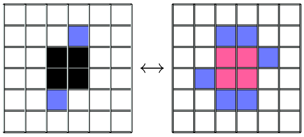


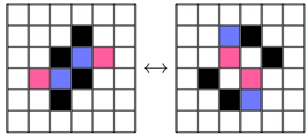
Figure 6. Solutions to GoL included in Figure 5.

3.4 Toad Type of Solution

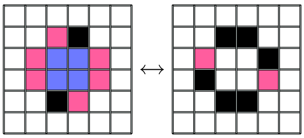
The next group of solutions include a “toad” of GoL for $a = 0$. Figure 7 shows the solutions and Figure 8 shows a toad of Gol.



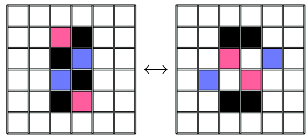
(a) toad ($a = 0$), block ($a = 1$)



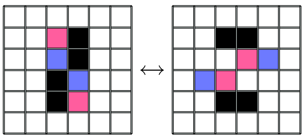
(b) toad ($a = 0$), barge ($a = 1$)



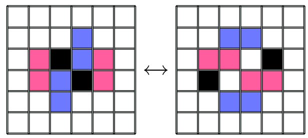
(c) toad ($a = 0$), pond ($a = 1$)



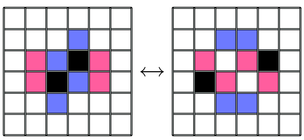
(d) toad ($a = 0$), snake ($a = 1$)



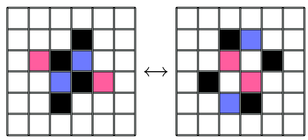
(e) toad ($a = 0$), snake ($a = 1$)



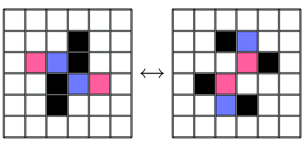
(f) toad ($a = 0$), snake ($a = 1$)



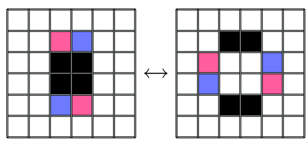
(g) toad ($a = 0$), snake ($a = 1$)



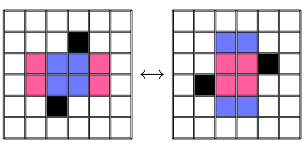
(h) toad ($a = 0$), clock ($a = 1$)



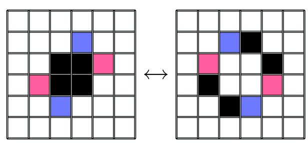
(i) toad ($a = 0$), clock ($a = 1$)



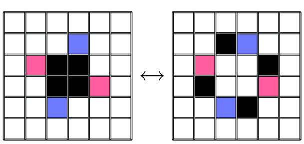
(j) toad ($a = 0$), toad ($a = 1$)



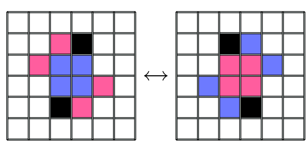
(k) toad ($a = 0$), toad ($a = 1$)



(l) toad ($a = 0$), toad ($a = 1$)



(m) toad ($a = 0$), toad ($a = 1$)



(n) toad ($a = 0$), toad ($a = 1$)

Figure 7. Solutions to equation (3) with period 2 giving a toad for $a = 0$. For $a = 1$, they give (a)–(g) a static solution, (h)–(i) a clock, and (j)–(n) another configuration of a toad.

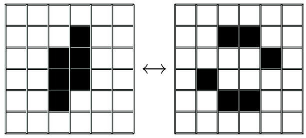


Figure 8. Toad of GoL.

3.5 Glider Type of Solution

There are solutions giving a moving pattern of GoL. One of the simplest solutions is called a "glider," shown in Figure 9. Figure 10 shows the real-valued solution and it coincides with that of Figure 9 if $a = 1$ and gives another glider of a different time phase if $a = 0$.

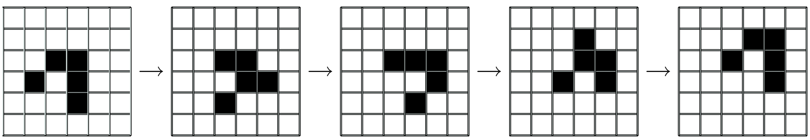


Figure 9. Glider of GoL.

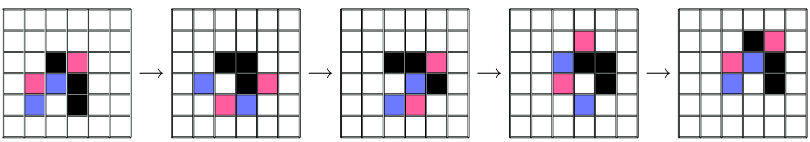


Figure 10. Solution to equation (3) giving a glider with a different time phase for $a = 0$ and 1.

3.6 More General Solution

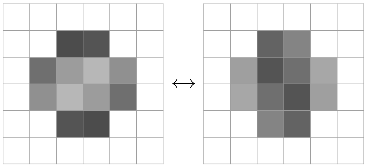
There are other variations of solutions obtained by rotating or reflecting those described in Sections 3.3 and 3.4. We can derive a general solution unifying all such solutions. Assume a periodic and symmetric solution with period 2 and with a point symmetry confined in a 4×4 region as shown here:

$$
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & u_{20}^0 & u_{10}^0 & 0 & 0 \\
0 & u_{31}^0 & u_{21}^0 & u_{11}^0 & u_{01}^0 & 0 \\
0 & \boxed{u_{01}^0} & \boxed{u_{11}^0} & \boxed{u_{21}^0} & \boxed{u_{31}^0} & 0 \\
0 & 0 & \boxed{u_{10}^0} & \boxed{u_{20}^0} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}
\leftrightarrow
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & u_{20}^1 & u_{10}^1 & 0 & 0 \\
0 & u_{31}^1 & u_{21}^1 & u_{11}^1 & u_{01}^1 & 0 \\
0 & \boxed{u_{01}^1} & \boxed{u_{11}^1} & \boxed{u_{21}^1} & \boxed{u_{31}^1} & 0 \\
0 & 0 & \boxed{u_{10}^1} & \boxed{u_{20}^1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

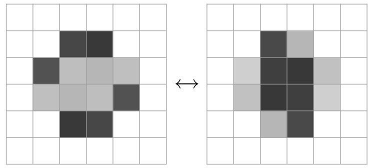
Since periodicity and point symmetry are assumed, only the boxed variables need to be determined. Then the general solution is given by 20 parameters a_i $(1 \le i \le 20)$ as follows:

$$
\begin{aligned}
u_{11}^0 &= a_1 + a_3 + a_5 + a_8 + a_9 + a_{12} + a_{17} + a_{18} + a_{19} + a_{20}, \\
u_{21}^0 &= a_1 + a_4 + a_6 + a_7 + a_{10} + a_{11} + a_{17} + a_{18} + a_{19} + a_{20}, \\
u_{01}^0 &= a_2 + a_4 + a_7 + a_8 + a_{10} + a_{12} + a_{13} + a_{15} + a_{16} + a_{20}, \\
u_{10}^0 &= a_2 + a_4 + a_5 + a_6 + a_9 + a_{11} + a_{13} + a_{14} + a_{16} + a_{17}, \\
u_{20}^0 &= a_2 + a_3 + a_5 + a_6 + a_{10} + a_{12} + a_{13} + a_{14} + a_{15} + a_{18}, \\
u_{31}^0 &= a_2 + a_3 + a_7 + a_8 + a_9 + a_{11} + a_{14} + a_{15} + a_{16} + a_{19}, \\
u_{11}^1 &= a_1 + a_3 + a_5 + a_8 + a_9 + a_{12} + a_{13} + a_{14} + a_{15} + a_{16}, \\
u_{21}^1 &= a_1 + a_4 + a_6 + a_7 + a_{10} + a_{11} + a_{13} + a_{14} + a_{15} + a_{16}, \\
u_{01}^1 &= a_2 + a_4 + a_7 + a_8 + a_9 + a_{11} + a_{16} + a_{17} + a_{19} + a_{20}, \\
u_{10}^1 &= a_2 + a_4 + a_5 + a_6 + a_{10} + a_{12} + a_{13} + a_{17} + a_{18} + a_{20}, \\
u_{20}^1 &= a_2 + a_3 + a_5 + a_6 + a_9 + a_{11} + a_{14} + a_{17} + a_{18} + a_{19}, \\
u_{31}^1 &= a_2 + a_3 + a_7 + a_8 + a_{10} + a_{12} + a_{15} + a_{18} + a_{19} + a_{20},
\end{aligned}
\tag{4}
$$

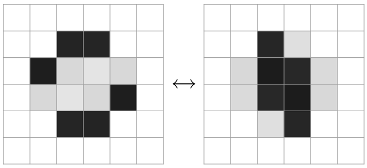
where $0 \le a_i \le 1$ for any i and $a_1 + a_2 + \cdots + a_{20} = 1$. If we set $a_i = 1$ and $a_j = 0$ $(j \ne i)$, one of the solutions reported in Sections 3.3 and 3.4 or its reflection or rotation is obtained. Note that 20 parameters are redundant, and we can reduce them to six through the transformation of parameters, though equation (4) is convenient to give a special solution. Figure 11 shows examples of solutions obtained by equation (4). Figure 11 (a), (b) and (c) show solutions for $a_{13} = 0.25$, 0.5 and 0.75, respectively, where other a_i are randomly given. We can observe that the toad solution to GoL emerges as a_{13} approaches 1.



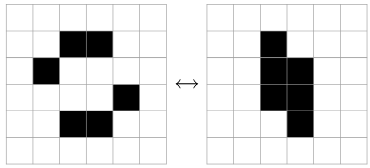
(a) $a_{13} = 0.25$



(b) $a_{13} = 0.5$



(c) $a_{13} = 0.75$



(d) $a_{13} = 1$

Figure 11. Examples of solutions obtained by equation (4).

4. MaxLife as a Multivalue Cellular Automaton

The range of u in equation (3) can be closed to the finite set $\{0, 1/N, 2/N, \ldots, (N-1)/N, 1\}$ for a positive integer N. Then equation (3) becomes an $(N+1)$-value CA. The case $N = 1$ is the original GoL. Figure 12 shows an example of evolution for 2-, 3- and 10-value cases ($N = 1, 2, 9$) with periodic boundary conditions from random initial data. Since it is difficult to evaluate the behavior of a general solution quantitatively, we describe our observation from numerical computation as shown in Figure 12. Solution of the 2-value case (GoL) tends to change drastically as time proceeds and often results in a steady state with separated static and periodic patterns. In contrast to the 2-value case, solution of the 3-or-more-value case rarely results in a steady state and continues to evolve with connected nonzero domains interacting with one another.

There are various basic static or periodic solutions confined in a finite region for the multivalue case. Some of them can be obtained by choosing the parameters of solutions reported in Section 3. For example, if a is set to $1/2$ for the solutions shown in Figures 3, 5, 7 and 10, they become solutions for the 3-value case, and if $1/3$, the 4-value case. Moreover, if the dimensions of the region for $u \neq 0$ and the period are assumed, all solutions can be searched numerically. For example, there are 40 static solutions and 23 periodic solutions with period 2 for the 3-value case within a 4×4 nonzero region. Solutions to GoL are included in them, and 13 of 40 static solutions and 3 of 23 periodic solutions are constructed only from 0 and 1. The number of steady basic solutions for the 3-value case is much larger than the

2-value case, and it suggests the persistence of evolution of the nonzero area for the multivalue case as shown in Figure 12.

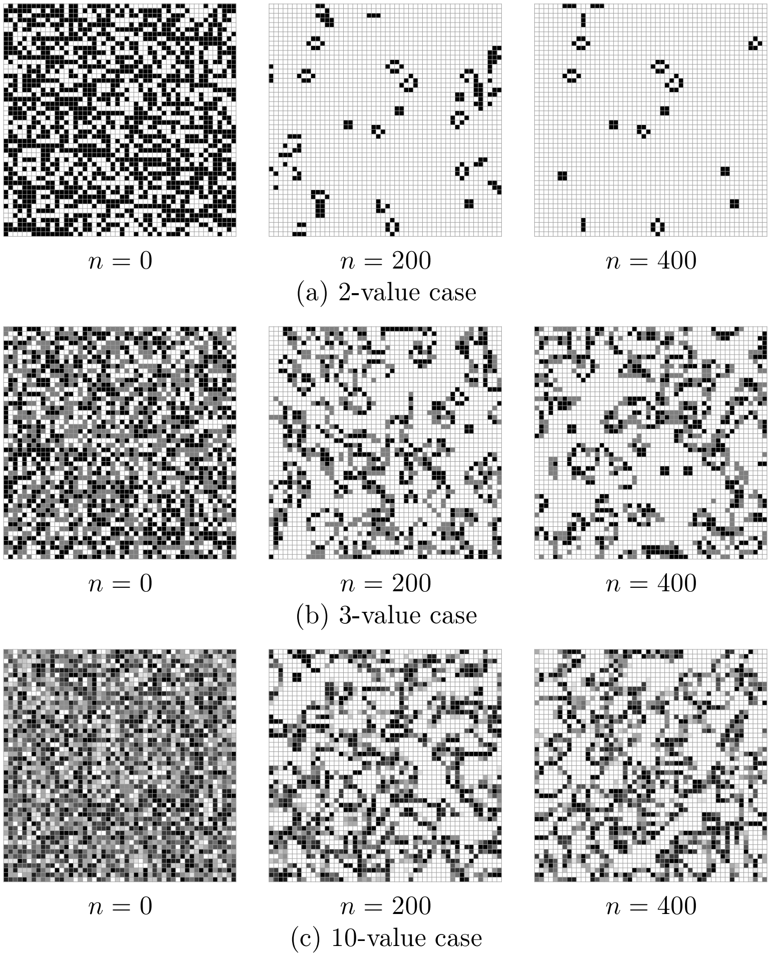


Figure 12. Evolution from random initial data for equation (3) as a multivalue CA.

5. Concluding Remarks

We proposed the max-plus equation (3) as the difference equation on a real-valued state variable. It includes the Game of Life (GoL) as a special case if the state value is restricted to 0 and 1. It has special solutions including a free parameter and they unify the solutions to GoL by special choice of parameter. Though various solutions to GoL have been reported independently, their relations are suggested through this unification. Among such solutions, we obtained a solution (equation (4)) including many parameters unifying various solutions to GoL. However, a systematic way to derive general solutions has not been found yet. One future problem is to propose a way to solve equation (3) as we solve differential equations.

The max-plus equation can be obtained from a difference equation using an exponential type of transformation of variables with a limiting parameter. Consider the following difference equation,

$$U_{ij}^{n+1} = C\frac{\left(1+\delta^2 U_{ij}^n S_{ij}^n\right)\left(1+\delta^4 S_{ij}^n\right)}{\left(1+\delta^3 U_{ij}^n S_{ij}^n\right)\left(1+\delta^3 S_{ij}^n\right)}, \tag{5}$$

where

$$S_{ij}^n = U_{i-1j-1}^n U_{ij-1}^n U_{i+1j-1}^n U_{i-1j}^n U_{i+1j}^n U_{i-1j+1}^n U_{ij+1}^n U_{i+1j+1}^n,$$

and

$$C = \frac{\left(1+\delta^3\right)^2}{\left(1+\delta^2\right)\left(1+\delta^4\right)}.$$

If we use the transformation including a parameter ε,

$$U_{ij}^n = e^{u_{ij}^n/\varepsilon}, \quad S_{ij}^n = e^{s_{ij}^n/\varepsilon}, \quad \delta = e^{-1/\varepsilon},$$

equation (3) is obtained from equation (5) by the limit $\varepsilon \to +0$. Note that the limiting procedure in equation (2) is used in the derivation.

An example of the evolution of the solution to equation (5) is shown in Figure 13 for $\varepsilon = 0.1$. The background value is $U = e^{0/\varepsilon} = 1$, and randomly chosen cells are set to $U = e^{1/\varepsilon}$ for the initial data. The initial data changes drastically at $n = 1$; some patterns survive and evolve from $n = 2$ to 16, and they merge and extend to the whole area from $n = 32$ to 128. This extended pattern evolves until at least $n = 10\,000$ and the range of U is always preserved from about 1 to $e^{1/\varepsilon}$. Though some stable static patterns confined in a finite area are found numerically, exact solutions giving static, periodic or moving patterns have not yet been found. It is another future problem to find exact solutions and to discuss the relation between solutions to equation (3) and to equation (5).

Figure 13. Examples of the evolution of the solution to equation (5) for $\varepsilon = 0.1$.

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