Some Quantum Mechanical Properties of the Wolfram Model

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This paper builds upon the techniques developed within our previous investigation of the relativistic and gravitational properties of the Wolfram Model—a new discrete spacetime formalism based on hypergraph transformation dynamics—in order to study classes of such models in which causal invariance is explicitly violated, as a consequence of non-confluence of the underlying rewriting system. We show that the evolution of the resulting multiway system, which effectively contains all possible branches of evolution history (corresponding to all possible hypergraph updating orders), is analogous to the evolution of a linear superposition of pure quantum eigenstates; observers may then impose "effective" causal invariance by performing a Knuth–Bendix completion operation on this evolution, thus collapsing distinct multiway branches down to a single, unambiguous thread of time, in a manner analogous to the processes of decoherence and wavefunction collapse in conventional quantum mechanics (and which we prove is compatible with a multiway analog of the uncertainty principle). By defining the observer mathematically as a discrete hypersurface foliation of the multiway evolution graph, we demonstrate how this novel interpretation of quantum mechanics follows from a generalized analog of general relativity in the multiway causal graph, with the Fubini–Study metric tensor playing the role of the spacetime metric, the quantum Zeno effect playing the role of gravitational time dilation and so on. We rigorously justify this correspondence by proving (using various combinatorial and order-theoretic techniques) that the geometry of the multiway evolution graph converges to that of complex projective Hilbert space in the continuum limit, and proceed to use this information to derive the analog of the Einstein field equations for the overall multiway system. Finally, we discuss various consequences of this "multiway relativity," including the derivation of the path integral, the derivation of particle-like excitations and their dynamics, the proof of compatibility with Bell's theorem and violation of the CHSH inequality, the derivation of the discrete Schrödinger equation and the derivation of the nonrelativistic propagator. Connections to many fields of mathematics and physics—including mathematical logic, abstract rewriting theory, automated theorem-proving, universal algebra, computational group theory, quantum information theory, projective geometry, order

https://doi.org/10.25088/ComplexSystems.29.2.537

theory, lattice theory, algorithmic complexity theory, advanced combinatorics, superrelativity, twistor theory and AdS/CFT-correspondence—are also discussed.

1. Introduction

In our previous paper [1], we formally introduced the Wolfram Model [2]—a new discrete spacetime formalism in which space is represented by a hypergraph, and in which laws of physics are modeled by transformation rules on set systems—and investigated its various relativistic and gravitational properties in the continuum limit, as first discussed in Stephen Wolfram's *A New Kind of Science* (NKS) [3]. Our central result was the proof that large classes of such models, with transformation rules obeying particular constraints, were mathematically consistent with discrete forms of both special and general relativity. An example of such a transformation rule is shown in Figure 1, and an example of its evolution is shown in Figures 2 and 3.

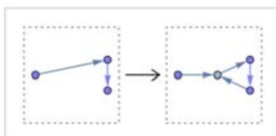


Figure 1. An example of a possible replacement operation on a set system, here visualized as a transformation rule between two hypergraphs (which, in this particular case, also happen to be equivalent to ordinary graphs). Adapted from [2].

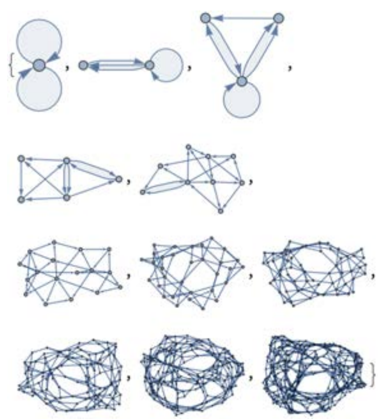


Figure 2. An example evolution of the above transformation rule, starting from an initial (multi)hypergraph consisting of a single vertex with two self-loops. Adapted from [2].

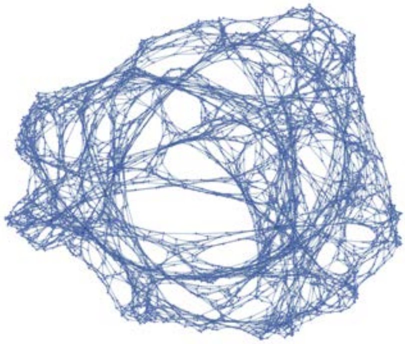


Figure 3. The final state of the above Wolfram Model evolution. Adapted from [2].

In particular, we introduced the notion of causal invariance (i.e., the condition that all causal graphs be isomorphic, independent of the choice of updating order for the hypergraphs), proved it to be equivalent to a discrete version of general covariance, with changes in updating order corresponding to discrete gauge transformations, and later used this fact to deduce discrete analogs of both Lorentz and local Lorentz covariance. Having derived the physical consequences of discrete Lorentz transformations in these models, we subsequently proved various results about the growth rates of volumes of spatial balls in hypergraphs and of spacetime cones in causal graphs, ultimately concluding that both quantities are related to discrete analogs of the Ricci curvature tensor for (hyper)graphs. We used this fact to prove that the condition that the causal graph should limit to a manifold of fixed dimensionality is equivalent to the condition that the discrete Einstein field equations are satisfied in the causal graph, and therefore that general relativity must hold. We went on to discuss some more speculative proposals regarding a general relativistic formalism for hypergraphs of varying local dimensionality and a few of the cosmological consequences that such a formalism would entail.

The present paper begins by briefly recapping the theory of abstract rewriting systems and their connections to the Wolfram Model in Section 2.1, before proceeding to introduce the Knuth–Bendix completion algorithm for "collapsing" distinct multiway evolution branches down to a single, unambiguous thread of time, thus obtaining effective causal invariance from a non-confluent rewriting system, in Section 2.2. We go on to show in Section 2.3 that the evolution of the multiway system is mathematically analogous to the evolution of a linear superposition of pure quantum eigenstates, and therefore that Knuth–Bendix completion is analogous to the process of decoherence and wavefunction collapse that occurs during the act of measurement within standard quantum mechanical formalism (indeed, we prove that this process is consistent with a multiway analog of the uncertainty principle). We also discuss some mathematical connections to universal algebra and computational group theory, as well as various implications of this new formalism for quantum information theory, in Section 2.4.

In Section 3.1 we introduce a new mathematical definition of a quantum observer as a discrete hypersurface foliation of the multiway evolution graph, and proceed to outline how the novel interpretation of quantum mechanics presented in the previous section therefore follows from a generalized variant of general relativity in the multiway causal graph, with the Fubini–Study metric tensor playing the role of the spacetime metric. We go on to prove this correspondence rigorously in Section 3.2, by first proving that the geometry of the multiway evolution graph converges to that of complex projective Hilbert space in the continuum limit (using various techniques from combinatorics, order theory and lattice theory, and by exploiting von Neumann's "continuous geometry" formalism for complex projective geometry), and then later by explicitly deriving the multiway variant of the Einstein field equations using the methods of superrelativity. Section 3.3 outlines a few geometrical and physical features of the multiway causal graph and makes a conjecture regarding its connection to the correspondence space of twistor theory. Finally, in Section 3.4 we discuss various consequences of "multiway relativity," including formal derivations of the path integral, the existence of particle-like excitations, the discrete Schrödinger equation, and the nonrelativistic propagator, as well as a proof of compatibility with Bell's theorem and the violation of the CHSH inequality.

2. Operators, Observables and the Uncertainty Principle

2.1 A Brief Recap: Abstract Term Rewriting, the Church–Rosser Property and Causal Invariance

Below, we will give a very terse recap of the mathematical formalism for hypergraph transformation rules and abstract rewriting systems and their relationship to multiway systems and causal invariance, as detailed much more comprehensively in [1, Section 2]. However, here we place a greater emphasis upon the logical and algebraic structure of rewriting systems and the relationships between the various notions of confluence [4–7].

Definition 1. An "abstract rewriting system" is a set, denoted A (in which each element of A is known as an "object" or an "element"), equipped with a binary relation, denoted $\to$, known as the "rewrite relation."

Definition 2. $\rightarrow^*$ designates the reflexive transitive closure of $\rightarrow$; that is, it represents the transitive closure of $\rightarrow \bigcup =$, where $=$ is the identity relation.

Definition 3. An object $a \in A$ is said to be "confluent" if:

$$\begin{aligned}&\forall\, b, c \in A,\ \text{such that } a \rightarrow^* b \text{ and } a \rightarrow^* c,\\ &\exists\, d \in A \text{ such that } b \rightarrow^* d \text{ and } c \rightarrow^* d.\end{aligned} \tag{1}$$

Definition 4. An abstract rewriting system A is said to be (globally) "confluent" (or otherwise to exhibit the "Church–Rosser property") if each of its objects $a \in A$ is itself confluent.

An illustration of a confluent object a being rewritten to yield objects b and c, which are subsequently made to converge back to some common object d, is shown in Figure 4.

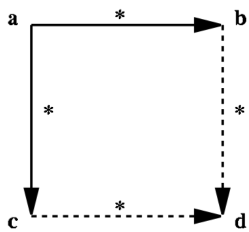
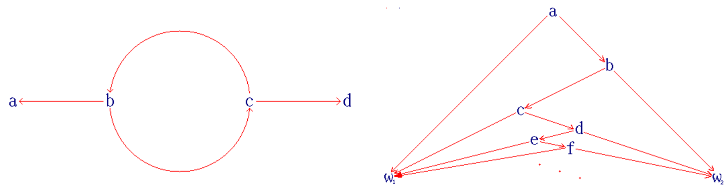


Figure 4. An illustration of a confluent object a being rewritten to yield distinct objects b and c; since a is confluent, b and c must be rewritable to some common object d, as shown. Image by Kilom691, distributed under a CC BY-SA 3.0 license.

Definition 5. An abstract rewriting system A is said to be "locally confluent" if objects that diverge after a single rewrite application must converge:

$$\begin{aligned}&\forall\, b, c \in A,\ \text{such that } a \rightarrow b,\ a \rightarrow c,\\ &\exists\, d \in A \text{ such that } b \rightarrow^* d \text{ and } c \rightarrow^* d.\end{aligned} \tag{2}$$

Local confluence is a strictly weaker property than global confluence; illustrations of two locally confluent rewriting systems that are not globally confluent are shown in Figure 5. The first evades global confluence by virtue of being cyclic; the second is acyclic but evades global confluence by virtue of the existence of an infinite rewrite sequence.



(a) (b)

Figure 5. (a) A locally confluent rewriting system, in which the presence of a cycle allows the system to evade global confluence. (b) A locally confluent and acyclic rewriting system, in which the presence of an infinite linear rewrite sequence allows the system to evade global confluence in essentially the same way. Images by Jochen Burghardt, distributed under a CC BY-SA 3.0 license.

Definition 6. An abstract rewriting system A is said to be "semi-confluent" if an object obtained by a single rewrite application and another object obtained by an arbitrary rewrite sequence are required to converge:

$$\begin{aligned} &\forall\, b, c \in A, \text{ such that } a \to b \text{ and } a \to^* c, \\ &\exists\, d \in A \text{ such that } b \to^* d \text{ and } c \to^* d. \end{aligned} \tag{3}$$

Note that an abstract rewriting system is semi-confluent if and only if it is globally confluent.

Definition 7. An abstract rewriting system A is said to be "strongly confluent" if, given two objects that diverge after a single rewrite application, one is required to converge with either zero or one rewrite application, and the other may converge by an arbitrary rewrite sequence:

$$\begin{aligned} &\forall\, b, c \in A, \text{ such that } a \to b \text{ and } a \to c, \\ &\exists\, d \in A \text{ such that } b \to^* d \text{ and } \left(c \to d \text{ or } c = d\right). \end{aligned} \tag{4}$$

Definition 8. An abstract rewriting system A is said to possess the "strong diamond property" if objects that diverge after a single rewrite application are required to converge after a single rewrite application:

$$\begin{aligned} &\forall\, b, c \in A, \text{ such that } a \to b \text{ and } a \to c, \\ &\exists\, d \in A \text{ such that } b \to d \text{ and } c \to d. \end{aligned} \tag{5}$$

Definition 9. A "hypergraph rewrite rule," denoted R, for a finite, undirected hypergraph $H = (V, E)$ [8]:

$$E \subseteq P(V) \backslash \{\emptyset\}, \tag{6}$$

is an abstract rewrite rule of the form $H_1 \to H_2$, in which H_1 denotes a subhypergraph of H, and H_2 denotes a new subhypergraph with identical symmetries and the same number of outgoing edges as H_1.

Definition 10. A "multiway system," denoted G_{multiway}, is a directed acyclic graph, corresponding to the evolution history of a (generally non-confluent) abstract rewriting system, in which each vertex corresponds to an object, and the edge $A \to B$ exists if and only if there exists a rewrite rule application that transforms object A to object B.

Examples of multiway systems corresponding to (globally) non-confluent and confluent substitution systems are shown in Figures 6 and 7, respectively; in the former case, we see that distinct rewriting paths from the same initial condition yield distinct normal forms (i.e., different eventual outcomes—results that cannot be rewritten further), while in the latter evolutions we see that the same eventual result is always obtained, independent of the choice of rewriting order.

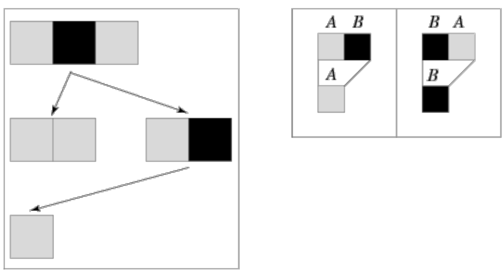


Figure 6. The multiway system generated by the evolution of a (globally) non-confluent string substitution system $\{AB \to A, BA \to B\}$ starting with the initial condition ABA, in which distinct normal forms are obtained for the two possible rewriting paths. Adapted from [3, p. 205].

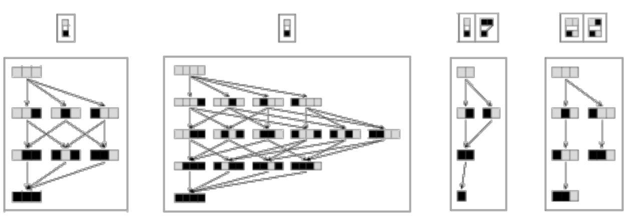


Figure 7. The multiway systems generated by the evolution of three (globally) confluent string substitution systems—the first two evolutions generated by $A \to B$, and the last two generated by $\{A \to B, BB \to B\}$ and $\{AA \to BA, AB \to BA\}$, respectively. Irrespective of which rewriting path is chosen, the same eventual outcome (i.e., the same normal form) is always reached. Adapted from [3, pp. 507 and 1037].

Definition 11. A "causal graph," denoted G_{causal}, is a directed acyclic graph, corresponding to the causal structure of an abstract rewriting system, in which every vertex corresponds to a rewrite rule application, and the edge $A \to B$ exists if and only if the rewrite application designated by B was only applicable as a consequence of the outcome of the rewrite application designated by A (i.e., the input for event B has a nontrivial overlap with the output for event A).

Definition 12. A multiway system is "causal invariant" if the causal graphs corresponding to all possible evolution histories eventually become isomorphic as directed acyclic graphs.

Examples of causal graphs generated by (globally) non-confluent and confluent substitution systems are shown in Figures 8 and 9, respectively.

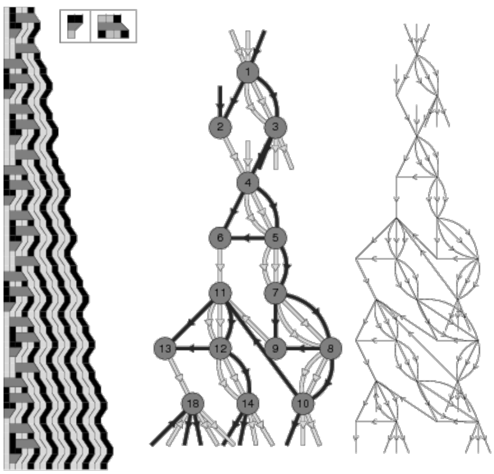


Figure 8. The causal graph generated by one possible evolution of a (globally) non-confluent string substitution system $\{BB \to A, AAB \to BAAB\}$; the combinatorial structure of the causal graph is ultimately dependent upon which branch of the multiway system, and therefore which of the possible rewriting orders, is followed. Adapted from [3, p. 498].

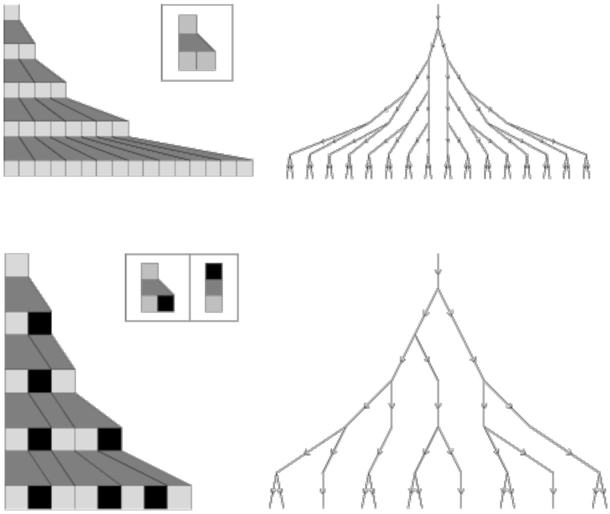


Figure 9. The causal graphs generated by the evolution of two (globally) confluent string substitution systems—$A \to AA$ and $\{A \to AB, B \to A\}$, respectively. The combinatorial structure of the causal graph is invariant to the choice of multiway system path, and therefore to the choice of rewriting order. Adapted from [3, p. 500].

2.2 Multiway Evolution and Knuth–Bendix Completion

One intuitive interpretation of the evolution of a multiway system for a non-causal invariant system, and therefore one in which distinct evolution branches can yield nonisomorphic causal graphs, is that the system is evolving according to *every possible* evolution history (i.e., all possible updating orders), any pair of which may have observationally distinct consequences. Such an interpretation brings forth strong connotations of the path integral formulation of quantum mechanics, in which the overall trajectory of a quantum system is taken to be described by a sum (or, more properly, a functional integral) over all possible trajectories, weighted by their respective amplitudes [9–11]:

$$\psi(x, t) = \frac{1}{Z} \int_{\mathbf{x}(0)=x} \mathcal{D}\mathbf{x} e^{iS[\mathbf{x}, \dot{\mathbf{x}}]} \psi_0(\mathbf{x}(t)), \tag{7}$$

where $\psi(x, t)$ denotes the overall wavefunction, $\psi_0(\mathbf{x}(t))$ denotes the wavefunction in the position representation, $\mathcal{D}\mathbf{x}$ denotes the integration measure over all possible paths starting at point $\mathbf{x}(0) = x$, Z is some arbitrary normalization constant, and S denotes the action:

$$S[\mathbf{x}, \dot{\mathbf{x}}] = \int dt \mathcal{L}(\mathbf{x}(t), \dot{\mathbf{x}}(t)). \tag{8}$$

Although it will not be proved formally until the next section, one of the purposes of the present paper is to demonstrate that this correspondence is not merely an analogy; more specifically, although the current section will focus primarily on the logical and algebraic structure of multiway systems, giving suggestive hints and making various conjectures regarding their connections to standard quantum mechanical formalism, the following section will show that the continuum limit of a non-causal invariant multiway system does indeed correspond to a quantum mechanical path integral in the usual sense.

As established in our previous paper [1], both special and general relativity depend ultimately upon the causal invariance of the underlying hypergraph transformation rules in order to hold globally across spacetime, which initially seems to be at odds with our requirement of non-causal invariance in our quantum mechanical interpretation of the multiway evolution. Indeed, we conjecture that this apparent incompatibility may be, at some deeper level, one of the fundamental sources of difficulty in the reconciliation of general relativity with quantum mechanics: relativistic evolution requires confluence, while quantum mechanical evolution requires non-confluence. The objective of the present subsection is to describe a rather subtle technique for circumventing this apparent incompatibility, by introducing and exploiting appropriately generalized versions of the critical pair lemma and Newman's lemma from mathematical logic [12], and the Knuth–Bendix completion algorithm from universal algebra [13–15].

Definition 13. Suppose that $\{x \to y, u \to v\}$ corresponds to a pair of (possibly identical) rules in some term rewriting system, with all variables renamed such that no variables are in common between the two rules. If x_1 is a subterm of x that is not a variable (x_1 could potentially be x itself), and if the pair (x_1, u) is unifiable, with the most general unifier θ, then the pair:

$$\langle y\theta, x\theta\{x_1\theta \to v\theta\}\rangle, \tag{9}$$

is known as a "critical pair" of that system.

Therefore, a critical pair arises in a term rewriting system (i.e., a rewriting system in which all objects are logical expressions containing nested subexpressions, as in the hypergraph case) whenever two rewrite rules "overlap" in such a way that two distinct objects can be obtained from a single common object. In a more general sense, critical pairs correspond to bifurcations in the evolution of a multiway system—in this general (non-term rewriting) context, we shall refer to them as "branch pairs." For instance, the term rewriting system:

$$\{f(g(x, y), z) \to g(x, z), g(x, y) \to x\}, \tag{10}$$

is associated with a nontrivial critical pair, namely:

$$\langle g(x, z), f(x, z)\rangle, \tag{11}$$

since both objects may be derived from the common expression $f(g(x, y), z)$, through the application of a single rewrite rule. On the other hand:

$$\langle f(x, x), f(x, x)\rangle \tag{12}$$

is a trivial critical pair for the system consisting of the single rule $f(x, y) \to x$, since this rule can be applied to the common expression $f(f(x, x), x)$ to yield $f(x, x)$ in two distinct ways.

Definition 14. A critical pair $\langle x, y\rangle$ of a term rewriting system A is said to be "convergent" if there exists some $z \in A$ such that $x \to^* z$ and $y \to^* z$.

The "critical pair lemma" is then the (more-or-less tautological) statement that a term rewriting system A is locally (weakly) confluent if and only if all of its critical pairs are convergent. In other words, one is able to determine algorithmically whether a given system is locally confluent, simply by computing all of its critical pairs and testing whether or not they converge (although such a procedure is, unsurprisingly given the undecidability of the word problem, not guaranteed to terminate in general). In the general case of a multiway evolution, we can say that a necessary condition of causal invariance is that all branch pairs generated by the multiway system eventually converge to some common successor state, such that no "disconnected" branches of the multiway graph are allowed to exist. Any apparently (locally) disconnected multiway branches may be interpreted as corresponding to a distinct superselection sector in projective Hilbert space, as will be made implicitly clear by our future derivations.

Definition 15. An object $a \in A$ is a "normal form" if there does not exist any $b \in A$ such that $a \to b$.

More informally, and as briefly mentioned earlier, a normal form is any object that cannot be rewritten further. For instance, for a term rewriting system consisting of a single rule $f(x, y) \to x$, the single element 4 would constitute a normal form of the object $f(f(4, 2), f(3, 1))$, since there exists the following rewrite sequence:

$$f(f(4, 2), f(3, 1)) \to f(4, 2) \to 4, \tag{13}$$

and as the rule does not apply to the single element 4, the resultant object can be rewritten no further.

Definition 16. An object $a \in A$ is said to be "weakly normalizing" if there exists a finite rewrite sequence such that $a \to^* b$, where b is a normal form.

Definition 17. An object $a \in A$ is said to be "strongly normalizing" if every finite rewrite sequence $a \to^* \ldots$ eventually terminates at some normal form.

Definition 18. A term rewriting system A is said to be "weakly normalizing" or "strongly normalizing" if each of its constituent objects is itself weakly normalizing or strongly normalizing, respectively.

For instance, the example presented above corresponds to a strongly normalizing system, since the length of an object strictly decreases with every application of the rule $f(x, y) \to x$, thus guaranteeing that there cannot exist any infinite rewrite sequences. On the other hand, the system:

$$\{f(x, y) \to x, f(x, x) \to f(3, x)\}, \tag{14}$$

is only weakly normalizing (since only objects that contain no $f(3, 3)$ are guaranteed to be strongly normalizing), and finally the system consisting of the single rule $f(x, y) \to f(y, x)$ is neither weakly nor strongly normalizing, since every rewrite sequence is infinite, for example:

$$f(1, 2) \to f(2, 1) \to f(1, 2) \to f(2, 1) \to \ldots. \tag{15}$$

"Newman's lemma," otherwise known as the "diamond lemma," states that if a term rewriting system A is strongly normalizing and locally (weakly) confluent, then it is also (globally) confluent. In fact, the full claim is slightly stronger: a strongly normalizing abstract rewriting system is (globally) confluent if and only if it is locally confluent. For the more general case of an arbitrary multiway evolution, Newman's lemma and the critical pair lemma together imply that, for a multiway system in which every branch eventually terminates, the convergence of all branch pairs is not only a necessary condition for causal invariance, but also a sufficient one.

Together, Newman's lemma and the critical pair lemma indicate that, if one is able to force all critical pairs to converge within some strongly normalizing rewriting system, without sacrificing the strong normalization property in the process, then doing so will force the resulting system to be (globally) confluent; this idea forms the abstract basis of the so-called "Knuth–Bendix completion" algorithm from universal algebra. The essential underlying concept of Knuth–Bendix completion is to compute the unresolved (i.e., non-convergent) critical pairs of the rewrite system $\langle x, y \rangle$ and then to add pairs of new rewrite rules of the form $\{x \to y, y \to x\}$, thereby imposing an effective equivalence relation between the two elements of the critical pair, in such a way as to force the critical pair to converge after a single step. The critical pair lemma guarantees that, so long as this process terminates, the resulting system will be locally confluent, and Newman's lemma guarantees that, so long as the process also preserves the strong normalization property, the system must accordingly be globally confluent.

Speaking more formally, given some set of equivalences between objects, denoted E (where each equivalence of the form $x = y$ can be thought of as corresponding to a pair of rewrite rules $\{x \to y, y \to x\}$), Knuth–Bendix completion is a semi-decision procedure for constructing a confluent and strongly normalizing term rewriting system R that shares the same "deductive closure" as E, in the following sense [16–18]:

Definition 19. Given a set of equivalences E, viewed as being a binary relation between objects, the "rewrite closure" of E, denoted $\to_E$, is the smallest rewrite relation that contains E.

Definition 20. Given a set of equivalences E, the "deductive closure" of E, denoted $\leftrightarrow_E^*$, is the equivalence closure of $\to_E$, that is, the smallest equivalence relation that contains $\to_E$.

As such, in the context of (for instance) a theorem-proving system, one may think of the deductive closure of E as being the set of all equivalences that can be derived as valid theorems by applying the equivalences from E in any order. A more general definition can be constructed for the non-equational case, that is, for sets of (potentially asymmetrical) rewrite relations:

Definition 21. Given a set of rewrite rules R, viewed as a binary relation, the "rewrite closure" of R, denoted $\to_R$, is the smallest rewrite relation that contains R.

Definition 22. Given a set of rewrite rules R, viewed as a binary relation, the "converse closure" of R, denoted $\leftarrow_R$, is the converse relation of $\to_R$, that is:

$$\begin{aligned} &\text{given}(\to_R) \subseteq X \times Y, \\ &(\leftarrow_R) = \{(y, x) \in Y \times X : (x, y) \in (\to_R)\}. \end{aligned} \tag{16}$$

Definition 23. Given a set of rewrite rules R, viewed as a binary relation, the "deductive closure" of R, denoted $\to_R^* \circ \leftarrow_R^*$, is the relation formed by the composition of the reflexive transitive closures of $\to_R$ and $\leftarrow_R$, denoted $\to_R^*$ and $\leftarrow_R^*$, respectively.

Thus, in a similar way to that described above, one can think of the deductive closure of R as being the set of all equivalences that can be derived as valid theorems by applying the rewrite rules from R (systematically, from left to right), until the two sides of the equivalence are literally equal.

For instance, if E is a set of equivalences corresponding to the standard identity, inverse and associativity axioms from group theory:

$$E = \{1 \cdot x = x,\, x^{-1} \cdot x = 1,\, (x \cdot y) \cdot z = x \cdot (y \cdot z)\}, \tag{17}$$

then the following group-theoretic theorem:

$$a^{-1} \cdot (a \cdot b) \leftrightarrow_E^* b, \tag{18}$$

is an element of the deductive closure of E, due to the existence of the following finite deduction chain:

$$a^{-1} \cdot (a \cdot b) \leftrightarrow_E^* (a^{-1} \cdot a) \cdot b \leftrightarrow_E^* 1 \cdot b \leftrightarrow_E^* b. \tag{19}$$

On the other hand, if one instead considers R, the corresponding set of (asymmetrical) rewrite rules:

$$R = \{1 \cdot x \to x,\, x^{-1} \cdot x \to 1,\, (x \cdot y) \cdot z \to x \cdot (y \cdot z)\}, \tag{20}$$

then although the following theorem:

$$(a^{-1} \cdot a) \cdot b \to_R^* \circ \leftarrow_R^* b \cdot 1, \tag{21}$$

is an element of the deductive closure of R, due to the existence of the following finite rewrite chain:

$$(a^{-1} \cdot a) \cdot b \to_R^* 1 \cdot b \to_R^* b \leftarrow_R^* b \cdot 1, \tag{22}$$

the analog of our original theorem:

$$a^{-1} \cdot (a \cdot b) \to_R^* \circ \leftarrow_R^* b, \tag{23}$$

is no longer such an element, since the asymmetry of the rewrite rules in R prevents one from applying a right-to-left variant of the rule:

$$(x \cdot y) \cdot z \to x \cdot (y \cdot z), \tag{24}$$

as required. A graphical depiction of how the Knuth–Bendix completion procedure can reduce the computational complexity of determining whether two expressions x and y are equivalent, by first expanding the rewrite relation to allow for the existence of unique normal forms for x and y, and then testing whether both expressions x and y reduce to a common normal form, is shown in Figure 10.

Therefore, a generalized multiway version of the Knuth–Bendix completion procedure allows one to start from some multiway system that is terminating (such that every branch eventually reaches some normal form after a finite number of steps), but whose evolution is not causal invariant, and from it to produce a new multiway evolution with the same deductive closure as the old one (i.e., in which the set of all equivalences between multiway states, when effectively "modded out" by the updating rules, is strictly preserved), but which is nevertheless causal invariant. In practice, this is actualized by adding new effective updating rules that force branch pairs resulting from the bifurcation of the multiway evolution to reconverge, thereby collapsing distinct branches of the multiway system back to a single effective evolution history. Our conjecture, which we formalize and prove in the following, is that this procedure is formally equivalent to wavefunction collapse in quantum mechanics (although the complete derivation of this apparent collapse from the multiway analog of quantum decoherence will not be presented until the following section).

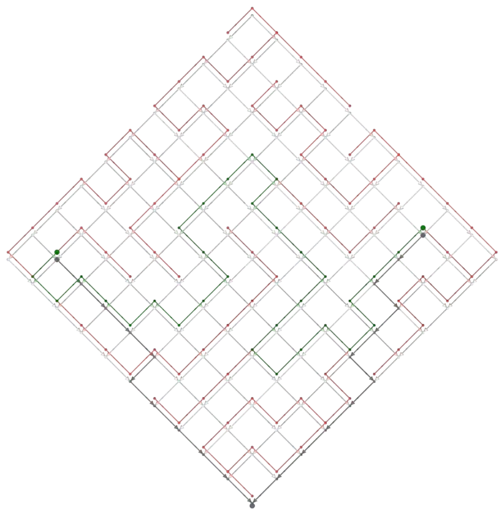


Figure 10. A graphical representation of the Knuth–Bendix completion procedure for proving an equational theorem; proving that $x \leftrightarrow^* y$ is computationally difficult, since it generally requires performing a heuristic search (shown in red) before a correct proof can be found (shown in green), whereas proving that x and y reduce to the same normal form, denoted $x\downarrow = y\downarrow$, is computationally easy (shown in gray). The Knuth–Bendix completion procedure has the effect of expanding the rewrite relation $\rightarrow$ so as to necessitate the existence of unique normal forms for x and y. Image by Jochen Burghardt, distributed under a CC BY-SA 4.0 license.

Rather excitingly, it is a standard consequence of term rewriting theory that, although Knuth–Bendix completion always preserves deductive closure, applying it can have the effect of allowing new states of the multiway system to become reachable, which were not previously obtainable using the non-causal invariant rule set; essentially, this occurs because the new rules that get added as a result of imposing equivalences between branch pairs can have slightly greater generality than the original set of updating rules. An immediate corollary of our conjecture is that these new states should correspond to quantum interference effects between neighboring branches of the multiway system, which we shall see explicitly in the forthcoming section.

2.3 Wavefunction Collapse and the Uncertainty Principle

As alluded to earlier, we can begin by thinking of states existing on distinct branches of the multiway system as corresponding to orthonormal eigenstates of the universe, with the overall evolution of the multiway system thereby corresponding to the evolution of some linear superposition of these eigenstates, that is, to the unitary evolution of a wavefunction. This is more than merely an analogy; as we shall prove rigorously in the following section, the geometry of the global multiway system converges to the geometry of a (complex) projective Hilbert space in the limit of an infinite number of updating events, thus allowing us to consider a "branchlike hypersurface" of states in the multiway system (i.e., a collection of multiway states that can be considered simultaneous with respect to some universal time function, analogous to a spacelike hypersurface in the relativistic case, and as defined formally later) as being an element of that projective Hilbert space. More precisely, such a hypersurface may be considered to be a linear superposition of the basis eigenstates of the multiway system [19]:

$$|\psi\rangle = \sum_i c_i \, |\phi_i\rangle, \tag{25}$$

where the c_i are probability amplitudes corresponding to each eigenstate $|\phi_i\rangle$ (in the multiway case, these amplitudes are concretely specified by a sum of the incoming path weights for the associated vertex in the multiway graph, where these path weights are computed using the discrete multiway norm, as defined below). By assumption, these eigenstates form an orthonormal basis for the Hilbert space:

$$\forall\, i, j, \quad \langle \phi_i \phi_j \rangle = \delta_{ij}, \tag{26}$$

where δ_{ij} denotes the standard Kronecker delta function, thus allowing us to interpret the hypersurface state $|\phi\rangle$ as being a normalized wavefunction for some generic quantum system:

$$\langle \psi \,|\, \psi \rangle = \sum_i |c_i|^2 = 1. \tag{27}$$

Knuth–Bendix completion therefore allows one to "collapse" the evolution of the multiway system, in such a way as to exhibit only a single global thread of effective evolution history; this is concretely achieved by constructing equivalence classes between distinct branches of the multiway system through the addition of new update rules that effectively impose equivalences between unresolved branch pairs. If we interpret the set $\{ | \phi_i \rangle \}$—the set of possible resolution states achievable via Knuth–Bendix completion—as being the eigenbasis of some observable operator, denoted $\hat{Q}$, applied to the multiway system, then we can interpret the completion procedure as being a measurement of the observable corresponding to $\hat{Q}$. Completion (i.e., observation) thus causes the wavefunction to "collapse" to a single eigenstate [20–22]:

$$| \psi \rangle \to | \phi_k \rangle, \quad \text{for some } k \in \mathbb{N}, \tag{28}$$

where the a priori probability of collapsing $| \psi \rangle$ to eigenstate $| \phi_k \rangle$ is given by the multiway Born rule for evolution path weights:

$$P_k = |\psi \phi_k|^2 = |c_k|^2; \tag{29}$$

it is meaningful to say that the wavefunction has "collapsed," because, a posteriori:

$$c_i = \delta_{ik}, \tag{30}$$

since, post-completion, there exists only a single effective branch of multiway evolution.

A rigorous mathematical explanation for why the act of observation should cause the multiway system to undergo a completion procedure is detailed in the next section (which first requires providing a formal mathematical definition of a multiway observer); however, an ad hoc intuitive explanation can be given presently in terms of coarse-graining procedures in statistical mechanics. Specifically, in classical statistical mechanics, the number of microstates for a given system is generally uncountably infinite, since the positions and momenta of particles generically take values in the real numbers $x_i, p_i \in \mathbb{R}$. Therefore, in order to be able to define quantities such as Ω, namely the number of microstates that are consistent with a given macrostate, within Boltzmann's entropy formula [23]:

$$S = k_B \log(\Omega), \tag{31}$$

it is first necessary to "coarse-grain" these microstates, essentially by grouping them together into equivalence classes, thereby obtaining a countable set of states (modulo equivalence). In order to define such an equivalence class, one defines two particles i and j as being in "equivalent" (which, in this particular case, means "observationally indistinguishable") states if their respective positions and momenta are sufficiently nearby in value, that is:

$$\exists\, \delta x, \delta p \in \mathbb{R}, \quad \text{such that } |x_i - x_j| < \delta x \text{ and } |p_i - p_j| < \delta p. \tag{32}$$

If every vertex in a multiway evolution graph corresponds to a distinct microstate of the universe, and if the branch pair $\langle x, y \rangle$ is found and forced to converge (by adding the pair of update rules $\{x \to y, y \to x\}$ via the Knuth–Bendix algorithm), then this completion step effectively corresponds to the definition of an equivalence relation between microstates x and y. Consequently, by analogy, each such branch pair completion step may be viewed as an enforcement of the statement that the microstates x and y are indistinguishable to any sufficiently macroscopic observer, as a result of that observer's naturally induced coarse-graining. The statement of correctness of the Knuth–Bendix algorithm can therefore be interpreted as the statement that, at least in certain cases, it is possible, given sufficient coarse-graining, to make the evolution of a quantum mechanical system appear macroscopically classical (i.e., causal invariant) to such observers.

In order to understand intuitively why a macroscopic observer should cause such a completion procedure to occur, we must first cease to view the observer as an abstract entity independent of the multiway evolution, and instead view them as an extended and persistent structure embedded within the multiway system itself. This leads to the following, somewhat philosophical, interpretation of what it means to be an observer in the context of a multiway evolution, which we shall make mathematically precise in the next section:

Definition 24. An "observer" is any persistent structure within a multiway system that perceives a single, definitive evolution history.

Loosely speaking, therefore, we can think of the observer as being any entity in the multiway system that "thinks" a definite sequence of events occurred: within their own internal representation of the world, the history of the universe must appear to be causal invariant. In order to be able to form such a coherent representation, the observer must themself have undergone a sufficient level of coarse-graining/branch pair completion to have their own internal representation be at least subjectively causal invariant (i.e., they must have undergone some form of Knuth–Bendix procedure, so as to ensure that at least all of their own local branch pairs converge, from their own subjective point of view).

If we now assume that the observer is sufficiently large that their internal hypergraph structure constitutes a statistically representative sample of the hypergraph structure of the rest of the universe, then the minimal set of branch pair completion rules (and hence, coarse-grained microstate equivalences) that would be required to make the observer themself be causal invariant will also cause the remainder of the universe to be causal invariant, at least from the vantage point of that particular observer. Therefore, the only requirements that an observer must fulfill in order to necessitate collapsing the evolution of the multiway system (and hence, the universal wavefunction) are to be sufficiently macroscopic and to possess a causal invariant internal representation of the world. For the remainder of this subsection, we will show how such an interpretation immediately results in the Wolfram Model satisfying a multiway form of the uncertainty principle. The formal proof of correctness of this interpretation is given in the next section.

In order to derive the multiway uncertainty principle, we begin by formally introducing the notion of commutation for rewrite relations in abstract rewriting systems [24]:

Definition 25. If $R_1 = \langle A, \to_1 \rangle$ and $R_2 = \langle A, \to_2 \rangle$ denote a pair of abstract rewriting systems, sharing the same object set A but with different rewrite relations $\to_1$ and $\to_2$, then R_1 and R_2 are said to "commute" if:

$$\begin{aligned} &\forall\, x, \quad x \to_1^* y \text{ and } x \to_2^* z \Longrightarrow \exists\, w, \\ &\quad \text{such that } y \to_2^* w \text{ and } z \to_1^* w. \end{aligned} \tag{33}$$

We can see immediately that an abstract rewriting system R is confluent if and only if it commutes with itself; a far less obvious observation was first made by Hindley and Rosen [25, 26], who formulated the so-called "commutative union theorem," which states that if

$$\forall\, i \in I, \quad R_i = \langle A, \to_i \rangle \tag{34}$$

denotes an indexed family of abstract rewriting systems, such that R_i commutes with R_j for all $i, j \in I$, then their union:

$$\bigcup_{i \in I} R_i = \left\langle A, \bigcup_{i \in I} \to_i \right\rangle, \tag{35}$$

is always confluent. Moreover, the so-called "commutativity lemma" gives a nontrivial sufficient condition for the commutativity of systems R_1 and R_2, namely:

$$\begin{aligned} &\forall\, x, \quad x \to_1 y \text{ and } x \to_2 z \Longrightarrow \exists\, w, \\ &\quad \text{such that } (y = w \text{ or } y \to_2 w) \text{ and } z \to_1^* w, \end{aligned} \tag{36}$$

which is directly related to the statement that strong confluence is a sufficient condition for (global) confluence.

An immediate general consequence of these results is that if the evolution of a multiway system is not causal invariant, then there will exist pairs of updating events that do not commute; the outcome of the evolution of a hypergraph (in the Wolfram Model case) will therefore depend upon the precise time ordering of the applications of these events, since the two distinct branches of the multiway system corresponding to the two distinct possible timelike orderings of these events will not, in general, reconverge on some common state. Given our interpretation of the multiway system as a discrete analog of (complex) projective Hilbert space, we may interpret rewrite relations as being linear operators $\hat{A}$ acting upon this space [27–29]:

$$\hat{A} = \sum_i e_i \hat{A}_i, \tag{37}$$

for basis vectors e_i, and with each component operator $\hat{A}_i$ yielding a corresponding eigenvalue a_i when applied to the wavefunction ψ:

$$\hat{A}_i \psi = a_i \psi, \tag{38}$$

where this wavefunction corresponds to a particular "branchlike hypersurface" in the multiway system, and the associated eigenvalue corresponds to the sum of path weights for a particular state or collection of states in some neighboring hypersurface (corresponding to the subsequent evolution step). Interpreting this rewrite relation as corresponding to some observable quantity A, it therefore follows that:

$$\hat{A} \psi = a \psi, \tag{39}$$

for some eigenvalue a, which itself corresponds to the measured value of A; this fact follows immediately from the linearity of $\hat{A}$:

$$\hat{A} \psi = \left(\sum_i e_i \hat{A}_i \right) \psi = \sum_i \left(e_i \hat{A}_i \psi \right) = \sum_i (e_i a_i \psi). \tag{40}$$

For a pair of such rewrite relations, $\hat{A}$ and $\hat{B}$, associated to observable quantities A and B, we can define their commutator as being the distance in the neighboring branchlike hypersurface (with respect to the multiway norm, defined in the next section) between the state obtained by applying first $\hat{A}$ and then $\hat{B}$, and the state obtained by applying first $\hat{B}$ and then $\hat{A}$:

$$\left[\hat{A}, \hat{B} \right] = \hat{A} \hat{B} - \hat{B} \hat{A}; \tag{41}$$

by linearity of $\hat{A}$ and $\hat{B}$, it follows that their commutator is itself a linear operator on our Hilbert space:

$$\left[\hat{A}, \hat{B} \right] \psi = \hat{A} \hat{B} \psi - \hat{B} \hat{A} \psi. \tag{42}$$

The commutator thus effectively quantifies the precision with which observables A and B can be measured simultaneously. More precisely, if ψ is considered an eigenfunction of both A and B, yielding respective eigenvalues a and b, then the statement that the rewrite relations associated to $\hat{A}$ and $\hat{B}$ commute, which we can now express as:

$$\left[\hat{A}, \hat{B} \right] \psi = 0, \tag{43}$$

implies that quantities A and B can be measured simultaneously with infinite precision, since, by linearity:

$$\left[\hat{A}, \hat{B}\right] = \hat{A}\hat{B}\psi - \hat{B}\hat{A}\psi = a\left(b\psi\right) - b(a\psi) = 0, \tag{44}$$

corresponding to the statement that the timelike ordering of commutative updating events (and hence, the time ordering of measurements on observables A and B) does not matter—irrespective of which quantity is measured first, observations of A will always yield a and observations of B will always yield b. Conversely, if the rewrite relations are non-commuting, which we can express as:

$$\left[\hat{A}, \hat{B}\right]\psi \neq 0, \tag{45}$$

then, as already established, the timelike ordering of the non-commutative updating events will have a macroscopic effect on the ultimate outcomes of those events in the multiway system, and hence the time ordering of the measurement operations will affect the outcomes of the two observations. In other words, measurements made of non-commuting observables necessarily take place on distinct branches of multiway evolution history [30]. This consequently places a limit on the precision with which observables A and B can be simultaneously prepared and measured:

$$\Delta A \Delta B \neq \left|\frac{\langle [A, B] \rangle}{2}\right|. \tag{46}$$

If one now assumes the unit distance on branchlike hypersurfaces to be given by $i\hbar$ (as will be justified formally in the next section), then this corresponds precisely to the statement of the uncertainty principle for rewrite relations on multiway systems:

$$\left[\hat{A}, \hat{B}\right] = i\hbar. \tag{47}$$

In particular, this allows us to interpret all pairs of single-step updating events $\hat{A}$ and $\hat{B}$ that do not commute as abstract rewriting systems as being canonically conjugate, and therefore we may consider the associated observables A and B to be Fourier transform duals of one another.

One additional elegant byproduct of this correspondence between updating events on multiway graphs and linear operators on projective Hilbert spaces is that it immediately renders many deep results of quantum field theory, such as the operator-state correspondence in the context of conformal field theory, exceptionally easy to prove. For instance, the formal statement of operator-state correspondence is that there exists a bijective mapping between operators, designated

$\phi(z, \bar{z})$, and states, designated $|\phi\rangle$, of the form:

$$|\phi\rangle = \lim_{z,\bar{z}\to 0} [\phi(z, \bar{z})] |0\rangle, \tag{48}$$

where $|0\rangle$ denotes the vacuum state, assumed to be invariant under the action of $SL(2, \mathbb{Z})$. In our generalized multiway context, this corresponds to the rather elementary statement that any state x in the multiway system can be constructed using an "ex-nihilo" updating event of the form:

$$\emptyset \to x. \tag{49}$$

2.4 Some Mathematical and Computational Considerations

In the context of universal algebra and computational group theory, the Knuth–Bendix completion algorithm used within the collapse of the multiway system wavefunction may be thought of as being a semi-decision procedure for solving the word problem for a specified algebra. More specifically, if $M = \langle X | R \rangle$ denotes some finitely presented monoid, with a finite generator set X and a finite relation set X, then the free monoid generated by X, denoted X^*, may be interpreted as the set of all words in X. The elements of M will now correspond to equivalence classes of X^* under an equivalence relation generated by R, and for each such class $\{w_1, w_2, \ldots\}$ there will exist some canonical representative word for that class, which we can denote w_k. This canonical word may be considered to be a normal form with respect to the term rewriting system associated to M, since the canonical element is presumed to be minimal with respect to some well-ordering relation, denoted $<$, satisfying the property of translation-invariance:

$$\forall A, B, X, Y \in X^*, \quad A < B \Longrightarrow XAY < XBY. \tag{50}$$

Therefore, since a confluent term rewriting system allows one to compute the unique normal form for each word $x \in X^*$, it also allows one to solve the word problem over M.

Once recast in this purely algebraic form, we can see that Buchberger's algorithm [31, 32] for determining properties of ideals in polynomial rings (and their associated algebraic varieties) is ultimately just a particular specialized instantiation of the Knuth–Bendix algorithm. More precisely, given a polynomial ring [33]:

$$R = K[x_1, \ldots, x_n], \tag{51}$$

over some field K, the ideal generated by F, where F denotes some finite set of polynomials in R:

$$F = \{f_1, \ldots, f_k\}, \tag{52}$$

corresponds to the set of linear combinations of elements in F, with coefficients given by elements of R:

$$\langle f_1, \ldots, f_k \rangle = \left\{ \sum_{i=1}^{k} g_i f_i : g_1, \ldots g_k \in K[x_1, \ldots, x_n] \right\}. \tag{53}$$

The individual monomials, which are themselves products of the form:

$$M = x_1^{a_1} \cdots x_n^{a_n}, \tag{54}$$

for some non-negative integers a_i, and which appear naturally in the construction of the polynomials in R:

$$c_1 M_1 + \cdots + c_m M_m, \quad \text{where } c_i \in K \text{ and } c_i \neq 0, \tag{55}$$

then have a total order defined upon them, which we can denote $<$. This total order is required to be compatible with multiplication, such that:

$$\forall\, M, N, P \text{ monomials}, \quad M \leq N \Longleftrightarrow MP \leq NP, \tag{56}$$

and:

$$\forall\, M, P \text{ monomials}, \quad M \leq MP, \tag{57}$$

thus yielding the translation-invariance property mentioned earlier. Buchberger's observation was then effectively that, given some ideal I in a polynomial ring, the so-called "Gröbner basis," denoted G, which is the generating set of I with the property that the ideal generated by the leading terms of polynomials in I is equal to the ideal generated by the leading terms in G (with respect to the aforementioned monomial order relation), can be computed by a procedure analogous to Knuth–Bendix completion.

This dual interpretation of the Knuth–Bendix procedure as a systematic method both for collapsing the multiway wavefunction and for solving the word problem for an arbitrary finitely presented monoid has significant computational implications regarding the interpretation of quantum information in multiway systems; more specifically, it is indicative of a fundamental equivalence between the collapse of a multiway evolution corresponding to a quantum computation, and the solution to the word problem for the monoid of unitary automorphisms of Hilbert space that constitutes the transition function for a three-tape quantum Turing machine. This has several profound implications for the relative computational power of classical Turing machines, quantum Turing machines and nondeterministic Turing machines, an example of which we will outline in the following.

A classical Turing machine, defined in terms of a partial transition function, is an entirely deterministic abstract machine [34]:

Definition 26. A one-tape "classical Turing machine," denoted:

$$M = \langle Q, \Gamma, b, \Sigma, \delta, q_0, F \rangle, \tag{58}$$

is a 7-tuple, consisting of a finite, nonempty set of states (denoted Q), a finite, nonempty alphabet (denoted Γ), a blank symbol (denoted $b \in \Gamma$), a set of input symbols (denoted $\Sigma \subseteq \Gamma \backslash \{b\}$), an initial state (denoted $q_0 \in Q$), a set of final states (denoted $F \subseteq Q$), and a partial function known as the "transition function," denoted:

$$\delta : (Q \backslash F) \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}, \tag{59}$$

where $\{L, R\}$ designates the set of possible shift directions for the tape head (with elements corresponding to left shift and right shift, respectively).

On the other hand, a nondeterministic Turing machine, by replacing this partial function with a partial relation (or multivalued partial function), allows instead for nondeterministic evolution [35]:

Definition 27. A one-tape "nondeterministic Turing machine" consists of the same 7-tuple as a classical Turing machine, but with the partial transition function now replaced with a partial transition relation:

$$\delta \subseteq ((Q \backslash F) \times F) \times (Q \times \Gamma \times \{L, S, R\}), \tag{60}$$

with an expanded set of shift directions $\{L, S, R\}$, now also allowing for no movement of the tape head.

Finally, a quantum Turing machine, rather than evolving along a single trajectory in a nondeterministic fashion, like a nondeterministic Turing machine, instead evolves a linear superposition of all possible trajectories [36–38]:

Definition 28. A three-tape "quantum Turing machine" (one tape for input, one for intermediate results and one for output—necessarily distinct in order to preserve unitarity of evolution) also consists of the same 7-tuple as a classical Turing machine, but with the set of states Q now replaced by a Hilbert space, the alphabet Γ replaced by another (potentially distinct) Hilbert space, the blank symbol $b \in \Gamma$ replaced by a zero vector, the input symbols $\Sigma \subseteq \Gamma \backslash \{b\}$ taken to be a discrete set (as in the classical case), the initial state $q_0 \in Q$ replaced by either an initial pure or mixed quantum state, the set of final states $F \subseteq Q$ replaced by a subspace of the original Hilbert space Q, and the transition function replaced with a transition monoid of the form:

$$\delta : \Sigma \times Q \otimes \Gamma \rightarrow \Sigma \times Q \otimes \Gamma \times \{L, R\}, \tag{61}$$

that is, some collection of unitary matrices corresponding to automorphisms of the underlying Hilbert space Q.

Therefore, if we interpret distinct branches of the multiway system as enacting computations, then we obtain the following, exceedingly clear picture of the relationship among these three formal notions of computation: a classical Turing machine is effectively evolving along a single definite path in the multiway system (using some deterministic rule to select which branch to follow), a nondeterministic Turing machine is also evolving along a single path (but now using a nondeterministic rule to select the sequence of successive branches), and a quantum Turing machine corresponds to the entire multiway evolution itself (i.e., to the superposition of all possible paths).

This leads to an immediate and potentially testable prediction of our interpretation of quantum mechanics: namely that following appropriate coarse-graining (i.e., following the application of a Knuth–Bendix completion), the class of problems that can be solved efficiently by quantum computers should be identical to the class of problems that can be solved efficiently by classical computers. More precisely, we predict in this appropriately coarse-grained case that $P = BQP$, where P and BQP denote the complexity classes of polynomial time and bounded error quantum polynomial time, respectively. Of course, this prediction presumes that the computational complexity of the completion procedure itself is negligible, which clearly is not true for multiway systems with sufficiently high rates of branch pair divergence; thus the prediction only holds for rules that are sufficiently "close" to causal invariance, in some suitably defined sense. The precise statement follows from the aforementioned interpretation because, although global multiway evolution in principle allows for potentially exponential (or even super-exponential) speedup over classical computations that make use of only a single branch, the act of measurement collapses the multiway system down to only a single branch of evolution history from the perspective of the observer. In other words, in order to maintain a causal invariant representation, the observer must perform a sufficient level of coarse-graining to ensure that any apparent advantage obtained through the use of a quantum computer over a classical one is effectively lost. This statement can be proved rigorously by applying the algebraic form of the Knuth–Bendix completion procedure to the finitely presented transition monoid of a quantum Turing machine, thereby reducing it to the partial transition function of a classical Turing machine, and therefore also reducing its set of final states to those that could have been generated in approximately the same number of evolution steps by using only a classical Turing machine. These computational complexity-theoretic consequences of our interpretation will be explored more fully in a future publication.

3. Branchlike Foliations and Multiway Relativity

3.1 Branchial Geometry and Multiway Invariance

One feature that is common to both multiway graphs and causal graphs is that their edges may be thought of as representing timelike separations, albeit in the multiway case those edges are updating events connecting timelike-separated global states of the universe, while in the causal case those edges are causal relations connecting timelike-separated updating events. However, one direct implication of this combinatorial correspondence is that, in much the same way as one is able to construct foliations of the causal graph into time-ordered sequences of discrete spacelike hypersurfaces (by effectively changing the order in which updating events get applied), one is also able to construct foliations of the multiway graph into time-ordered sequences of discrete "branchlike" hypersurfaces (by effectively changing the order in which branch pairs diverge). Given the theory of measurement developed in the previous section, a particular choice of branchial hypersurface may be interpreted as corresponding to a particular choice of microstates that the observer is assuming to be equivalent—a statement that we will be able to make precise momentarily.

As in the purely causal case described in [1], we can formalize this notion of "branchlike separation" by constructing a layered graph embedding of the multiway graph into the discrete "multiway lattice" $\mathbb{Z}^{1,n}$, for some (generically very large) n, and thus labeling the global multiway states by:

$$\mathbf{q} = (t, \mathbf{y}), \tag{62}$$

where $t \in \mathbb{Z}$ is the usual discrete time coordinate from the causal case, and:

$$\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{Z}^n, \tag{63}$$

are discrete "branchial" coordinates (i.e., the multiway analog of a spatial coordinate system). One subsequently induces a geometry on the embedded multiway graph in the standard way:

Definition 29. The discrete "multiway norm" is given by:

$$\| (t, \mathbf{y}) \| = \| \mathbf{y} \|^2 - t^2, \tag{64}$$

or, in more explicit form:

$$\| (t, \mathbf{y}) \| = \left(y_1^2 + \cdots + y_n^2\right) - t^2. \tag{65}$$

Definition 30. Global multiway states $\mathbf{q} = (t, \mathbf{y})$ are classified as "timelike," "entanglementlike" or "branchlike" based upon their discrete multiway norm:

$$\mathbf{q} \sim \begin{cases} \text{timelike}, & \text{if } \|(t, \mathrm{y})\| < 0, \\ \text{entanglementlike}, & \text{if } \|(t, \mathrm{y})\| = 0, \\ \text{branchlike}, & \text{if } \|(t, \mathrm{y})\| > 0. \end{cases} \tag{66}$$

Definition 31. Pairs of global multiway states $\mathbf{q} = (t_1, \mathrm{y}_1)$, $\mathbf{r} = (t_2, \mathrm{y}_2)$ can be classified as "timelike-separated," "entanglementlike-separated" or "branchlike-separated," accordingly:

$$(\mathbf{q}, \mathbf{r}) \sim \begin{cases} \text{timelike-separated}, \\ \qquad \text{if}((t_1, \mathrm{y}_1) - (t_2, \mathrm{y}_2)) \sim \text{ timelike}, \\ \text{entanglementlike-separated}, \\ \qquad \text{if}((t_1, \mathrm{y}_1) - (t_2, \mathrm{y}_2)) \sim \text{ entanglementlike}, \\ \text{branchlike-separated}, \\ \qquad \text{if}((t_1, \mathrm{y}_1) - (t_2, \mathrm{y}_2)) \sim \text{ branchlike}. \end{cases} \tag{67}$$

Our justification for referring to the intermediate form of multiway state separation as being "entanglementlike" is quite simply that when a global state A in the multiway system diverges to yield a branch pair (B, C), one can think of the states B and C as being "entangled" in the sense that, though they may not be causally connected, they are nevertheless now correlated by virtue of their multiway descendence from the common ancestor state A. The entanglement rate defined by the combinatorial structure of the multiway graph therefore determines which particular collections of multiway states are, in principle, able to influence which other collections of multiway states; by this token, the analog of a light cone in the causal graph is an "entanglement cone" in the multiway graph, which, by probing the entanglement structure of the multiway system, allows one effectively to determine the maximum theoretical rate of entanglement between global multiway states. Denoting the embedded multiway graph by $(\mathcal{M}, g)$, with some metric g defined by the multiway norm (the precise details of how to define an appropriate metric on an arbitrary multiway system are given in the following subsection), we consequently have (c.f. [39]):

Definition 32. A global state x "evolutionarily precedes" global state y, denoted $x \ll y$, if there exists a future-directed (i.e., monotonic downward) evolutionary (i.e., timelike) path through the multiway graph connecting x and y.

Definition 33. A global state x "strictly entanglementwise precedes" global state y, denoted $x < y$, if there exists a future-directed (i.e., monotonic downard) entanglement (i.e., non-branchlike) path through the multiway graph connecting x and y.

Definition 34. A global state x "entanglementwise precedes" global state y, denoted $x \prec y$, if either x strictly entanglementwise precedes y, or $x = y$.

Definition 35. The "evolutionary future" and "evolutionary past" of a global state x, denoted $I^{+}(x)$ and $I^{-}(x)$, are defined as the sets of global states that x evolutionarily precedes, and that evolutionarily precede x, respectively:

$$I^{+}(x) = \left\{y \in \mathcal{M} : x \ll y\right\}, \quad I^{-}(x)\{y \in M : y \ll x\}. \tag{68}$$

Definition 36. The "entanglement future" and "entanglement past" of a global state x, denoted $J^{+}(x)$ and $J^{-}(x)$, are defined as the sets of global states that x entanglementwise precedes, and that entanglementwise precede x, respectively:

$$J^{+}(x) = \left\{y \in \mathcal{M} : x \prec y\right\}, \quad J^{-}(x) = \left\{y \in \mathcal{M} : y \prec x\right\}. \tag{69}$$

By analogy to the purely relativistic case, we can see that $I^{+}(x)$ designates the interior of the future entanglement cone of x, while $J^{+}(x)$ designates the entire future entanglement cone, including the cone boundary itself. As expected, the definition of evolutionary and entanglement future and past can be extended to sets of global states $S \subset \mathcal{M}$ in the usual way:

$$I^{\pm}(S) = \bigcup_{s \in S} I^{\pm}(x), \quad J^{\pm}(S) = \bigcup_{x \in S} J^{\pm}(x). \tag{70}$$

This new formalism allows us to define precisely what kind of mathematical structure the previously introduced "branchlike hypersurfaces" actually are—namely, they correspond to the multiway analog of discrete Cauchy surfaces in a causal graph. The statement that the multiway graph can be foliated into a non-intersecting set of such hypersurfaces, that is, the analog of strong hyperbolicity for multiway graphs, is conjectured to be related to the unitarity of evolution in quantum mechanics. As discussed in the previous section, since each branch pair in the multiway graph can effectively be thought of as denoting a pair of possible outcomes for a measurement operation being applied to some global state (where the state being measured is the common ancestor of the branch pair, and the measurement operation corresponds to the rewrite relation being applied), it follows that possible foliations of the multiway graph correspond to different possible orderings in the application of measurement events, with each such ordering corresponding to a possible observer (or, more correctly, to a global equivalence class of observers with the same sequence of measurement choices). Finally, this allows us to provide a formal mathematical definition for what it means to be an observer in a multiway system:

Definition 37. An "observer" in a multiway system is any ordered sequence of non-intersecting branchlike hypersurfaces Σ_t that covers the entire multiway graph, with the ordering defined by some universal time function:

$$t : \mathcal{M} \to \mathbb{Z}, \qquad \text{such that } t \neq 0 \text{ everywhere}, \tag{71}$$

such that the branchlike hypersurfaces are the level sets of this function, satisfying:

$$\begin{aligned} &\forall\, t \in \mathbb{Z}, \Sigma_{t_1}\{p \in \mathcal{M} : t(p) = t_1\}, \\ &\quad \text{and } \Sigma_{t_1} \bigcap \Sigma_{t_2} = \emptyset \Longleftrightarrow t_1 \neq t_2. \end{aligned} \tag{72}$$

Therefore, the analog of a Lorentz transformation in the multiway case is any parameterized change of observer, with the notion of causal invariance (and hence, Lorentz covariance) now replaced with the analogous notion of "multiway invariance":

Definition 38. A multiway system is "multiway invariant" if the ordering of timelike-separated measurement events in the multiway graph is preserved under parameterized changes of the observer (i.e., under distinct choices of branchlike hypersurface foliation), even though the ordering of branchlike-separated measurement events is not.

Enforcing the constraint of multiway invariance also finally gives us a mathematically rigorous justification for our previously outlined interpretation of quantum measurement in terms of branch pair completions applied to the multiway evolution (although a more geometrical version of the same argument, in terms of the trajectories of geodesics in projective Hilbert space, is also given explicitly below). Specifically, each branchlike hypersurface can be thought of as designating an equivalence class between branchlike-separated multiway states (which are themselves elements of branch pairs, and assumed to be observationally indistinguishable by a macroscopic observer), and therefore a multiway-invariant system can be obtained by forcing all branch pairs on each such hypersurface to be convergent, for instance, by adding completion rules. In much the same way as discrete forms of special and general relativity can be derived from the principle of causal invariance, our central conjecture in this paper is that discrete forms of quantum mechanics and quantum field theory can be derived from the principle of multiway invariance. Since Newman's lemma guarantees that critical pair convergence implies confluence for terminating rewriting systems, it follows that multiway invariance implies causal invariance for any non-infinite multiway evolution.

However, there is still one key mathematical subtlety that we have glossed over, but which we must now return to in order to complete this proof. Namely, both the multiway graph and the causal graph are, at some level, approximations to a more fundamental object—the "multiway causal graph"—which contains edges corresponding to causal relations between all updating events, not only for a single evolution history (as defined by a single multiway branch), but across all possible evolution histories. Therefore, we can see that the structure of the multiway graph corresponds to the coarse, large-scale structure of the multiway causal graph (with the fine details of individual causal relations removed), whereas the structure of a purely relativistic causal graph is given by the fine structure of a single "bundle" of edges in the multiway causal graph—as we will see later, this is conjectured to be deeply related to the concept of twistor correspondence in twistor theory. Therefore, the true multiway norm is actually the discrete norm defined over the entire multiway causal graph, to which the multiway norm described earlier is only a coarse approximation. But a key difficulty immediately presents itself, since there are now three distinct types of separation between multiway causal vertices: spacelike separation, timelike separation and branchlike separation. To distinguish between the two separation distances in the causal graph, we simply denote timelike distances with negative numbers and spacelike distances with positive ones, but now branchlike distances necessitate a new numerical direction in order to differentiate them from spacelike distances, so (without loss of generality) we choose to use imaginary numbers.

The rigorous justification for choosing imaginary numbers for denoting branchlike distances comes from Sylvester's "law of inertia" for quadratic forms [40]; if A is a symmetric matrix defining some quadratic form, then, for every invertible matrix S such that:

$$D = SAS^{\intercal} \tag{73}$$

is diagonal, the number of negative (respectively, positive) elements of D is always the same. In other words, for a real quadratic form Q in n variables, expressed in diagonal form:

$$Q(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} a_i x_i^2, \quad a_i \in \{0, 1, -1\}, \tag{74}$$

the number of coefficients of Q of a given sign remains invariant under changes of basis; geometrically, this implies that the dimension of the subspace on which the form Q is positive definite (respectively, negative definite) is always the same. A generalization of this theorem, due to Ikramov [41], extends to any normal matrices A and B satisfying the property that:

$$B = SAS^{*}, \tag{75}$$

for some nonsingular complex matrix S, and states that A and B are congruent if and only if the number of eigenvalues on each open ray in $\mathbb{C}$ is the same for each. This result immediately implies that in order to maintain adequate discrimination among branchlike, spacelike and timelike separation across all possible observers, the metric tensor for the multiway causal graph must have eigenvalues with at least three distinct signs, hence necessitating the use of complex numbers. This has implications for the proof that branchlike hypersurfaces converge to complex projective Hilbert spaces in the limit of infinitely large multiway graphs, which we present in Section 3.2. In particular, the "projectivity' of the projective Hilbert space (i.e., the invariance of quantum states under rescaling by an arbitrary complex number) effectively arises from the additional redundant degree of freedom introduced by allowing for both positive and negative imaginary numbers, when only one such sign is strictly required.

As such, we can see that the true multiway norm for a multiway causal graph embedded in the discrete "multiway-Minkowski lattice" $\mathbb{Z}^{1,n,m}$, for $n, m \in \mathbb{N}$, with global events labeled by:

$$\mathbf{p} = (t, \mathbf{x}, \mathbf{y}), \tag{76}$$

for discrete spatial and branchial coordinates $\mathbf{x}$ and $\mathbf{y}$:

$$\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{Z}^n, \quad \mathbf{y} = (y_1, \ldots, y_m) \in \mathbf{Z}^m, \tag{77}$$

can now be defined:

Definition 39. The discrete "multiway-Minkowski norm" is given by:

$$\| (t, \mathbf{x}, \mathbf{y}) \| = \| \mathbf{x} \|^2 + i \| \mathbf{y} \|^2 - (1+i) t^2, \tag{78}$$

or, in more explicit form:

$$\| (t, \mathbf{x}, \mathbf{y}) \| = \left(x_1^2 + \cdots + x_n^2\right) + i\left(y_1^2 + \cdots + y_m^2\right) - (1+i) t^2. \tag{79}$$

This yields an associated classification of global events and their separations:

Definition 40. Global events $\mathbf{p} = (t, \mathbf{x}, \mathbf{y})$ are classified as "timelike," "lightlike," "entanglementlike," "spacelike" or "branchlike," based upon their discrete multiway-Minkowski norm:

$$\mathbf{p} \sim \begin{cases} \text{timelike}, & \text{if } \Re(\| (t, \mathbf{x}, \mathbf{y}) \|) < 0 \\ & \quad \text{and } \Im(\| (t, \mathbf{x}, \mathbf{y}) \|) < 0, \\ \text{lightlike}, & \text{if } \Re(\| (t, \mathbf{x}, \mathbf{y}) \|) = 0, \\ \text{entanglementlike}, & \text{if } \Im(\| (t, \mathbf{x}, \mathbf{y}) \|) = 0, \\ \text{spacelike}, & \text{if } \Re(\| (t, \mathbf{x}, \mathbf{y}) \|) > 0, \\ \text{branchlike}, & \text{if } \Im(\| (t, \mathbf{x}, \mathbf{y}) \|) > 0. \end{cases} \tag{80}$$

Definition 41. Pairs of global events $\mathbf{p} = (t_1, \mathbf{x}_1, \mathbf{y}_1)$, $\mathbf{q} = (t_2, \mathbf{x}_2, \mathbf{y}_2)$ can be classified as "timelike-separated," "lightlike-separated," "entanglementlike-separated," "spacelike-separated" and "branchlike-separated," respectively:

$$(\mathbf{p}, \mathbf{q}) \sim \begin{cases} \text{timelike-separated}, \\ \qquad \text{if}((t_1, \mathbf{x}_1, \mathbf{y}_1) - (t_2, \mathbf{x}_2, \mathbf{y}_2)) \sim \text{timelike}, \\ \text{lightlike-separated}, \\ \qquad \text{if}((t_1, \mathbf{x}_1, \mathbf{y}_1) - (t_2, \mathbf{x}_2, \mathbf{y}_2)) \sim \text{lightlike}, \\ \text{entanglementlike-separated}, \\ \qquad \text{if}((t_1, \mathbf{x}_1, \mathbf{y}_1) - (t_2, \mathbf{x}_2, \mathbf{y}_2)) \sim \text{entanglementlike}, \\ \text{spacelike-separated}, \\ \qquad \text{if}((t_1, \mathbf{x}_1, \mathbf{y}_1) - (t_2, \mathbf{x}_2, \mathbf{y}_2)) \sim \text{spacelike}, \\ \text{branchlike-separated}, \\ \qquad \text{if}((t_1, \mathbf{x}_1, \mathbf{y}_1) - (t_2, \mathbf{x}_2, \mathbf{y}_2)) \sim \text{branchlike}. \end{cases} \tag{81}$$

Note that there is no mutual exclusivity among lightlike-separated, entanglementlike-separated, spacelike-separated and branchlike-separated updating events; in particular, spacelike separation is, in general, a special case of branchlike separation, since the application of a pair of purely spacelike-separated updating events will usually yield a branch pair in the multiway graph (although this branch pair is always guaranteed to converge if the system is causal invariant, since causal invariance is a strictly necessary condition for multiway invariance). This seemingly rather trivial observation has the welcome consequence of immediately implying the de Broglie hypothesis (i.e., wave-particle duality) [42] within our formalism, for the following reason: while, as we have seen previously, a geodesic bundle in the causal graph can be interpreted as the collective trajectory of a set of test particles, a geodesic bundle in the multiway causal graph can be interpreted as the evolution of a wave packet (as will be demonstrated formally below). Therefore, the lack of mutual exclusivity between separation types makes it impossible for an observer in the multiway causal graph to determine whether a particular geodesic bundle corresponds to the evolution of a collection of test particles, a wave packet, or both (since the geodesics themselves will appear to be purely spacelike separated, purely branchlike separated, or some combination of the two, depending upon the observer's particular choice of multiway causal foliation). Thus, we see that wave-particle duality is just a specific consequence of the general principle of *multiway relativity*—a concept that we define and explore more fully below.

3.2 Complex Projective Hilbert Space and Multiway Relativity

Our principal goal for this subsection is to prove rigorously the claim upon which many of our previous arguments have implicitly relied: namely that, in the continuum limit of an infinitely large multiway system, the branchlike hypersurfaces embedded within a particular foliation of the multiway graph correspond to complex projective Hilbert spaces [43]. Subsequently, we show how a theory of "multiway relativity" can be derived as a consequence of the principle of multiway invariance, in much the same way as special and general relativity can be derived from the principle of causal invariance, with the natural metric on projective Hilbert space (namely, the Fubini–Study metric [44, 45]) taking the role of the standard spacetime metric tensor from general relativity. We thus demonstrate how all of the quantum mechanical principles discussed in this paper may be deduced from the mathematical analog of general relativity in multiway causal space and discuss some of the salient geometrical features of this space.

The first step in the proof of the limit of branchlike hypersurfaces to complex projective Hilbert spaces is noting that the natural (combinatorial metric) distance between a pair of vertices in such a hypersurface is indirectly a measure of their branch pair ancestry distance; if they are both elements of the same branch pair (i.e., if both follow directly from a common state via a single updating event), then they are separated by a distance of one, and otherwise they are separated by the number of levels back in the multiway hierarchy that one must traverse in order to find their common ancestor state. Disconnections in a branchial hypersurface therefore indicate a lack of common ancestry between states (which generally only happens if there was more than one initial condition for the system, or if the hypersurface grew faster than the maximum rate of entanglement). In other words, branchlike separation is, as one would intuitively expect, a measure of the degree of entanglement between pairs of global multiway states. Clearly, this distance is related to the edit distance (or Levenshtein distance [46]) metric on strings, but with a single "edit" here defined as a pair of replacement operations (one corresponding to a traversal up a branch pair, and the other corresponding to a traversal down). This can ultimately be formalized as a measure of algorithmic complexity [47], in terms of the minimum number of replacement operations required to derive the two states from some common ancestor state; as such, the natural distance on branchlike hypersurfaces is equivalent, at least up to some multiplicative constant, to the information distance metric from algorithmic complexity:

Definition 42. The "information distance" between two states x and y, denoted $ID(x, y)$, is defined by:

$$ID(x, y) = \min\{|p| : p(x) = y \wedge p(y) = x\}, \tag{82}$$

where p designates a finite program for some fixed universal Turing machine, accepting x and y as finite input states.

In our particular case, the instruction set of this Turing machine is assumed to correspond to the set of rewrite relations for the multiway system. The information distance metric can easily be seen to be a natural extension of the Kolmogorov algorithmic complexity measure [48], since [49]:

$$ID(x, y) = E(x, y) + O\big(\log(\max\{K(x \mid y), K(y \mid x)\})\big), \tag{83}$$

where:

$$E(x, y) = \max\{K(x \mid y), K(y \mid x)\}, \tag{84}$$

and $K(\cdot \mid \cdot)$ the standard Kolmogorov complexity. The distance function defined by $E(x, y)$ therefore satisfies all of the necessary axioms of a metric on the space of states, up to the following additive term in the metric inequalities [50]:

$$O\big(\log(\max\{K(x \mid y), K(y \mid x\})). \tag{85}$$

As discussed at length in our previous paper, the natural generalization of the notion of a volume measure on a Riemannian manifold to arbitrary metric-measure spaces (including both graphs and hypergraphs as special cases) is that of a probability measure, defined on some common probability space. The standard continuum limit of the discrete information distance metric for a statistical manifold (i.e., a Riemannian manifold whose points are all probability measures) corresponds to the so-called "Fisher information metric," that is, the canonical method of quantifying information distance between probability measures [51, 52]:

Definition 43. The "Fisher information metric" tensor, for a statistical manifold with coordinates:

$$\theta = (\theta_1, \theta_2, \ldots, \theta_n), \tag{86}$$

is given by:

$$g_{jk}(\theta) = \int_X \frac{\partial \log(p(x, \theta))}{\partial \theta_j} \frac{\partial \log(p(x, \theta))}{\partial \theta_k} p(x, \theta)\, dx, \tag{87}$$

for local coordinate axes j and k, where the integral is evaluated over all points $x \in X$, with X denoting some (either discrete or continuous) random variable, and $p(x)$ corresponding to some probability distribution that has been normalized as a function of θ:

$$\int_X p(x, \theta) dx = 1. \tag{88}$$

Fisher information distance can also be expressed in a pure matrix form as:

$$[I(\theta)]_{j,k} = \mathbb{E}\left[\left(\frac{\partial \log(p(x, \theta))}{\partial \theta_j}\right)\left(\frac{\partial \log(p(x, \theta))}{\partial \theta_k}\right)\middle|\, \theta\right]; \tag{89}$$

however, its correspondence with the previous (discrete) case of information distance is made manifest by rewriting the metric tensor in the alternative form:

$$g_{jk}(\theta) = \int_X \frac{\partial^2 i(x, \theta)}{\partial\theta_j \partial\theta_k} p(x, \theta) dx = \mathbb{E}\left[\frac{\partial^2 i(x, \theta)}{\partial\theta_j \partial\theta_k}\right], \tag{90}$$

where $i(x, \theta)$ denotes the standard entropy (i.e., self-information) contribution from classical information theory:

$$i(x, \theta) = -\log(p(x, \theta)). \tag{91}$$

It is also a well-known result that, if one extends from the case of Riemannian manifolds to the case of the complex projective Hilbert space $\mathbb{CP}^n$, defined by the space of all complex lines in $\mathbb{C}^{n+1}$:

$$\mathbb{CP}^n = \left\{\mathbf{Z} = [Z_0, Z_1, \dots, Z_n] \in \mathbb{C}^{n+1} \backslash \{0\}\right\} / \left\{\mathbf{Z} \sim c\mathbf{Z} : c \in \mathbb{C}^*\right\}, \tag{92}$$

that is, the quotient of $\mathbb{C}^{n+1} \backslash \{0\}$ by the diagonal group action of the multiplicative group:

$$\mathbb{C}^* = \mathbb{C} \backslash \{0\}, \tag{93}$$

corresponding to the equivalence relation of all complex multiples of points, then the Fisher information metric itself extends to become the so-called "Fubini–Study metric," that is, the natural Kähler metric on projective Hilbert spaces.

Definition 44. The "Fubini–Study metric" on the complex projective Hilbert space $\mathbb{CP}^n$, in terms of the homogeneous coordinates (i.e., the standard coordinate notation for projective varieties in algebraic geometry):

$$\mathbf{Z} = [Z_0, \dots, Z_n], \tag{94}$$

is defined by the line element:

$$ds^2 = \frac{|\mathbf{Z}|^2 |d\mathbf{Z}|^2 - \left(\overline{\mathbf{Z}} \cdot d\mathbf{Z}\right)\left(\mathbf{Z} \cdot d\overline{\mathbf{Z}}\right)}{|\mathbf{Z}|^4}, \tag{95}$$

or, in a more explicit form:

$$ds^2 = \frac{Z_\alpha \overline{Z}^\alpha dZ_\beta d\overline{Z}^\beta - \overline{Z}^\alpha Z_\beta dZ_\alpha d\overline{Z}^\beta}{\left(Z_\alpha \overline{Z}^\alpha\right)^2}. \tag{96}$$

It is worth noting at this point that, in the context of conventional formalism, the Fubini–Study metric is the natural metric induced by the geometrization of quantum mechanics; when defined in terms of pure states of the form [53, 54]:

$$| \psi \rangle = \sum_{k=0}^{n} Z_k \mid e_k \rangle = [Z_0 : Z_1 : \cdots : Z_n], \tag{97}$$

where $\{ \mid e_k \rangle \}$ designates any orthonormal basis set for the Hilbert space, the metric line element can be written in the following compact infinitesimal form:

$$ds^2 = \frac{\langle \delta\psi \mid \delta\psi \rangle}{\langle \psi \mid \psi \rangle} - \frac{\langle \delta\psi \mid \psi \rangle \langle \psi \mid \delta\psi \rangle}{\langle \psi \mid \psi \rangle^2}. \tag{98}$$

The Fubini–Study metric is correspondingly used in the quantification of state entanglement and other key properties; when considered in terms of mixed states, with associated density matrices ρ_1 and ρ_2, the Fubini–Study metric is easily shown to be equivalent to the so-called "quantum Bures metric," D_B [55, 56]:

$$D_N(\rho_1, \rho_2)^2 = 2\left(1 - \sqrt{F(\rho_1, \rho_2)}\right), \tag{99}$$

with the fidelity function F given by:

$$F(\rho_1, \rho_2) = \left[\mathrm{Tr}\left(\sqrt{\sqrt{\rho_1}\, \rho_2 \sqrt{\rho_1}} \right) \right]^2, \tag{100}$$

at least up to a multiplicative constant.

Having shown that the natural discrete metric on branchlike hypersurfaces converges to the Fisher information metric in the continuum limit, and that the Fisher information metric itself extends to the Fubini–Study metric for complex projective Hilbert spaces, all that remains is to show why branchlike hypersurfaces should yield *projective* Hilbert spaces in the continuum limit, as opposed to regular non-projective ones (i.e., ones equipped with the standard flat metric). As already alluded to in the previous section, the origin of the complex values in the multiway state vectors lies in the three distinct separation types in the multiway causal graph, and the origin of the projectivity of the associated Hilbert space arises from the additional degree of freedom introduced by having four distinct eigenvalue signs thanks to the use of arbitrary complex numbers. We can make this statement mathematically precise by considering the formal analogy between the entanglement structure of a multiway graph and the conformally invariant (i.e., causal) structure of a causal graph; the combinatorial structures of both causal graphs and multiway graphs are, crucially, invariant under the conformal transformation:

$$\hat{g} = \Omega^2 g, \tag{101}$$

for some conformal scaling factor Ω. In the case of a multiway graph, such a conformal rescaling is equivalent to modding out by the action of the multiplicative group $\mathbb{C}^*$, which completes the proof.

This idea that complex projective Hilbert spaces might be representable as the limiting case of some purely combinatorial structure, such as a branchlike hypersurface, is not entirely without precedent. For instance, at least superficially, it might appear to be related to the standard use of so-called combinatorial "Steiner systems" as a finite model for projective planes [57–59]:

Definition 45. A "Steiner system," denoted $S(t, k, v)$, is a set X consisting of v points, together with a collection of subsets of X known as "blocks," with each block being of size k, and with the constraint that any set of t distinct points in X is contained in exactly one such block.

The total number of blocks b is therefore given by:

$$b = \frac{v r}{k}, \tag{102}$$

with the constraint that $v \leq b$, and the number of blocks containing a given point r is given by:

$$r = \frac{\binom{v-1}{t-1}}{\binom{k-1}{t-1}}, \tag{103}$$

with the constraint that $k \leq r$. Then, in the particular case in which:

$$v = n^2 + n + 1, \quad k = n + 1, \quad t = 2, \tag{104}$$

and where all of the blocks are interpreted as lines, the Steiner system yields a discrete model for the finite projective plane, as shown in Figure 11 for the case of the Fano plane (i.e., the finite projective plane of order 2) [60].

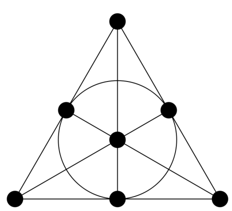


Figure 11. The Fano plane represented using the Steiner system $S(2, 3, 7)$ (historically known as a Steiner triple system, since $k = 3 = t + 1$); the seven blocks correspond to the seven lines of the plane, each of which contains exactly three points, and with the property that every pair of points lies on exactly one line.

However, while a Steiner system may be considered to be an "incidence-geometric" combinatorial model of a projective space, in which vertices correspond directly to points and edges correspond directly to lines, in such a way as to reflect the incidence structure of the projective plane, branchlike hypersurfaces constitute something more akin to a lattice-theoretic model for projective Hilbert space. The key distinction between the two approaches is that incidence geometry captures the relative locations of objects in projective space, whereas lattice theory captures their relative containment. More precisely, if we consider an order-theoretic lattice (i.e., a partially ordered set in which every pair of elements has a unique supremum known as a "join," denoted $\vee$, and a unique infimum known as a "meet," denoted $\wedge$) [61], then we can introduce the following self-duality condition on pairs of elements (a, b) [62]:

$$\forall\, x, \quad a \leq b \Longrightarrow a \vee (x \wedge b) = (a \vee x) \wedge b, \tag{105}$$

which may be thought of as being a weakened form of the distributivity condition, known as "modularity." A "modular lattice" is then a lattice in which every pair of elements satisfies the modularity condition, an example of which is shown in Figure 12.

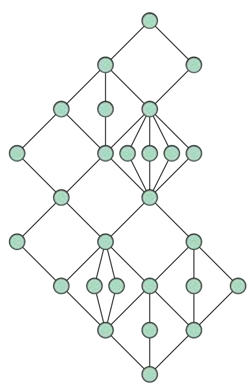


Figure 12. The Hasse diagram for a modular lattice in which the partial order may be represented as the intersection of two total orders (i.e., the lattice has order dimension 2).

It is known that a lattice is modular if and only if the "diamond isomorphism theorem" (the lattice-theoretic equivalent of the second isomorphism theorem) holds for each pair of elements; namely, if ϕ and ψ denote order-preserving maps on the intervals $[a \wedge b, b]$ and $[a, a \vee b]$, for some pair of elements (a, b), of the form:

$$\phi : [a \wedge b, b] \to [a, a \vee b], \quad \psi : [a, a \vee b] \to [a \wedge b, b], \tag{106}$$

defined by:

$$\phi(x) = x \vee a, \quad \psi(y) = y \wedge b, \tag{107}$$

then the diamond isomorphism theorem is said to hold for the pair (a, b) if and only if ϕ and ψ are isomorphisms of these intervals. In particular, the order-preserving composition:

$$\psi\phi : [a \wedge b, b] \to [a \wedge b, b], \tag{108}$$

will always satisfy the inequality:

$$\psi(\phi(x)) = (x \vee a) \wedge b \geq x, \tag{109}$$

with equality if and only if the pair (a, b) is modular; by the property that the dual of a modular lattice is always modular, the reverse composition:

$$\phi\psi : [a, a \vee b] \to [a, a \vee b], \tag{110}$$

is also an identity map, proving that ϕ and ψ must themselves be isomorphisms. An example of the satisfaction and failure of the diamond isomorphism theorem for modular and non-modular pairs, respectively, is shown in Figure 13.

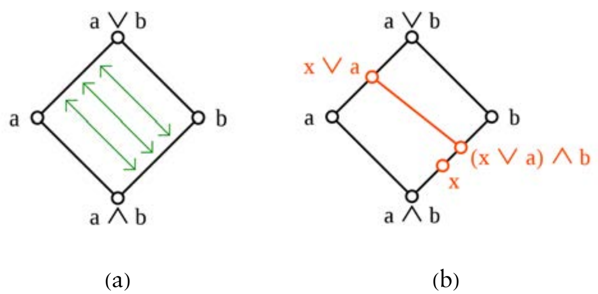


Figure 13. (a) We see the diamond isomorphism theorem being satisfied for the modular pair (a, b), since the maps $\phi : [a \wedge b] \to [a, a \vee b]$ and $\psi : [a, a \vee b] \to [a \wedge b, b]$ indicated by the green arrows are both bijective and order-preserving, and hence isomorphisms. (b) We see the failure of the diamond isomorphism theorem for the non-modular pair (a, b), since the map between $(x \vee a) \wedge b$ and $x \vee b$ is no longer invertible for some x, as shown by the orange line.

Thus, since a multiway evolution graph may be considered to be the Hasse diagram for a partially ordered set of global states (with the partial order defined by the observer's universal time function), we see that the diamond isomorphism theorem illustrates explicitly the correspondence between modular pairs in the multiway lattice and branch pairs that converge in a single step within the multiway evolution. Therefore, the modularity condition for lattices is equivalent to the strong diamond property for rewriting systems, and hence modularity of the multiway lattice constitutes a provably sufficient condition for causal invariance. The significance of this observation is that modularity is also one of the foundational properties of classical projective geometry; in our context, perspectives correspond to isomorphisms between intervals, and projections correspond to join-preserving maps. This can also be seen as a consequence of the diamond isomorphism theorem, which allows one to interpret the modularity condition in terms of projections onto the sublattice given by $[a, b]$.

These results provide the mathematical justification for the interpretation of the limiting structure of the multiway graph as being a complex projective space, thanks to the framework of "continuous geometry" developed by von Neumann [63], which weakens the standard axioms for complex projective geometry given by Menger and Birkhoff [64, 65] in terms of lattices of linear subspaces of the projective space. More specifically, any lattice satisfying the properties of modularity, completeness, continuity, complementarity and irreducibility may be interpreted as a continuous geometry, and therefore as a lattice-theoretic model of complex projective space. In this context, completeness refers to the lattice-theoretic property that all subsets of the lattice should possess both a join and a meet; continuity refers to the following associativity property of the join and meet operations:

$$\left(\bigwedge_{\alpha \in A} a_{\alpha}\right) \vee b = \bigwedge_{\alpha \in A} \left(a_{\alpha} \vee b\right), \tag{111}$$

and:

$$\left(\bigvee_{\alpha \in A} a_{\alpha}\right) \wedge b = \bigvee_{\alpha \in A} \left(a_{\alpha} \wedge b\right), \tag{112}$$

for some directed set (i.e., a join semilattice) A, and where the indices are defined such that $a_{\alpha} < a_{\beta}$ when $\alpha < \beta$; complementarity refers to the statement that every element a possesses a (not necessarily unique) complement b:

$$\forall\, a, \quad \exists\, b, \text{ such that } a \wedge b = 0 \text{and} a \vee b = 1, \tag{113}$$

where 0 and 1 denote the minimal and maximal elements of the lattice, respectively; and finally, irreducibility refers to the condition that the only two elements possessing unique complements are 0 and 1 themselves.

With the geometry of the multiway space thus established, the remainder of this paper will be dedicated to our central conjecture: that the mathematical formalisms of quantum mechanics and quantum field theory can ultimately be derived as the analogs of special and general relativity for multiway graphs, with branchlike hypersurfaces taking the role of spacelike hypersurfaces, and the Fubini–Study metric tensor taking the place of the standard (Riemannian) spacetime metric tensor. More concretely, our conjecture can be phrased as the statement that, in much the same way as general relativity places constraints on the trajectories of geodesic bundles in the causal graph, geometrical quantum mechanics places constraints on the trajectories of geodesic bundles (corresponding to collections of pure quantum states) in the multiway graph. The geometrical intuition behind this claim is that the multiway analogs of the Ricci and Weyl curvature tensors enforce conditions on the volumes and shapes of wave packets (i.e., multiway geodesic bundles), respectively, in much the same way as they do for causal geodesic bundles in the purely relativistic case.

Such a formulation of geometrical quantum mechanics has previously been considered by Leifer [66] in the context of developing the theory of "superrelativity," in which the geodesic motion of pure states corresponds to deformations of the "polarization ellipsoid" (i.e., the superrelativistic terminology for our notion of a wave packet), given by transformations from the pure state coset:

$$\frac{SU(n)}{S\left[U(1) \otimes (n-1)\right]}, \tag{114}$$

otherwise known as Goldstone modes, or alternatively to unitary rotations of the ellipsoid, given by transformations from the pure state isotropy group:

$$U(1) \otimes U(n-1), \tag{115}$$

otherwise known as Higgs modes. With pure states interpreted, as before, as being rays in the projective Hilbert space $\mathbb{CP}^{n-1}$:

$$|\psi\rangle = A \exp(i\alpha) \sum_{a=0}^{n-1} \psi^{\alpha} \mid a, x\rangle, \tag{116}$$

there consequently exists a natural choice of local coordinates $\pi^{i}_{(n)}$ for the $\mathbb{CP}^{n-1}$ chart atlas:

$$U_b = \left\{ \mid \psi\rangle = \sum_{a=0}^{n-1} \psi^a \mid a, x\rangle : \psi^b \neq 0 \right\}, \tag{117}$$

which, in the particular case of $b = 0$, becomes:

$$\pi^i_{(0)} = \frac{\psi^i}{\psi^0}, \tag{118}$$

for $1 \leq i \leq n-1$. In this specific instance, the Fubini–Study metric tensor on $\mathbb{CP}^n$ may be written in the following form:

$$g_{ik*} = 2\hbar \frac{\left(1 + \sum_{s=1}^{n-1} |\pi^s|^2\right)\delta_{ik} - \pi^{i*}\pi^k}{\left(1 + \sum_{s=1}^{n-1} |\pi^s|^2\right)^2}. \tag{119}$$

From here, one is also able to account for the modulus of the wave-function by effectively imposing a mild breaking of the projective symmetry, thus obtaining an appropriately generalized variant of Fubini–Study metric tensor:

$$G_{ik*} = 2\hbar R^2 \frac{\left(R^2 + \sum_{s=1}^{n-1} |\pi^s|^2\right)\delta_{ik} - \pi^{i*}\pi^k}{\left(R^2 + \sum_{s=1}^{n-1} |\pi^s|^2\right)^2}, \tag{120}$$

with R corresponding to the radius of the "density sphere":

$$\sum_{a=1}^{n-1} |\psi^a|^2 = R^2, \tag{121}$$

from which the rays in the ordinary Hilbert space $\mathbb{C}^n$ have been stereographically projected. Our central conjecture, just as that of superrelativity, is that this tensor plays the same role in geometric quantum mechanics as the Einstein curvature tensor plays in general relativity.

Note that the natural connection on the space $\mathbb{CP}^{n-1}$, namely:

$$\Gamma^i_{kl} = -2 \frac{\delta^i_k \pi^{l*} + \delta^i_l \pi^{k*}}{R^2 + \sum_{s=1}^{n-1} |\pi^s|^2}, \tag{122}$$

now defines an intrinsic gauge potential on multiway space, which we may interpret (in the case of a Wolfram Model hypergraph substitution system) as defining a fiber bundle over each spatial hypergraph. Concretely, for each vertex in a spatial hypergraph, there are many possible orientations in which a hypergraph replacement rule could be applied to that vertex (in general, there will be one such orientation for each hyperedge incident to the vertex), and we may interpret each such orientation as corresponding to a particular choice coordinate basis (i.e., some local section of a fiber bundle), which will subsequently place constraints on the set of possible orientations for other purely spacelike-separated rule applications. Thus, we can interpret the hypergraph itself as corresponding to some base space, with each vertex corresponding to a fiber, such that local gauge invariance in the multiway evolution therefore follows as an inevitable consequence of causal invariance of the underlying replacement rules.

3.3 Interpretations of Multiway Invariance and the Multiway Causal Graph

In much the same way as c can be thought of as being the effective separation between updating events yielded by distinct causal edges (since elementary light cones in the causal graph determine the maximum effective rate of information propagation), so too can $\hbar$ be thought of as being the effective separation between global states yielded by distinct multiway edges (since the elementary entanglement cones in the multiway graph determine the maximum effective rate of entanglement between global states, and therefore Planck's constant may be interpreted as being a measure of this maximum theoretical rate of quantum entanglement). This interpretation formally justifies our previous assumption that pairs of single non-commuting update events $\hat{A}$, $\hat{B}$ may be considered to be canonically conjugate:

$$\left[\hat{A}, \hat{B}\right] = i\hbar, \tag{123}$$

since each such pair of single-step events corresponds to a branch pair in the multiway graph, and therefore designates the effective distance between pairs of adjacent points in a branchlike hypersurface. It also explains the appearance of an angle quantity $e^{iS[\mathbf{x},\dot{\mathbf{x}}]}$ in the path integral formula:

$$\psi(x, t) = \frac{1}{Z} = \int_{\mathbf{x}(0)=x} \mathcal{D}\mathbf{x} e^{iS[\mathbf{x},\dot{\mathbf{x}}]} \psi_0(\mathbf{x}(t)), \tag{124}$$

since we can now see that this quantity designates the dispersion angle of geodesic bundles propagating through the multiway graph; as this angle is trivially related to the density of causal edges (via the density of updating events) in the multiway causal graph, we can deduce that it must consequently have units of energy, thus endowing the elementary distance in the multiway graph, that is, Planck's constant, with units of action (as required), and hence finally justifying (at least dimensionally) our interpretation of the multiway graph as a path integral in the first place.

The interpretation of $\hbar$ as corresponding to a maximum rate of entanglement between quantum states, and therefore as constituting some kind of fundamental "quantum speed limit" on the evolution of these states, is also a natural feature of the standard formalism of quantum mechanics. One can formally prove the existence of such a limit using the methods of Margolus and Levitin [67], by considering the expansion of some initial state $| \psi_0 \rangle$ in the energy eigenbasis:

$$| \psi_0 \rangle = \sum_n c_n | E_n \rangle, \tag{125}$$

with the solution to the time-dependent Schrödinger equation (assuming a constant Hamiltonian $\hat{H}$) at time t thus being given by:

$$| \psi_t \rangle = \sum_n c_n \exp\left(-i \frac{E_n t}{\hbar}\right) | E_n \rangle. \tag{126}$$

The "overlap" between the initial and final states is therefore given by a time-dependent function $S(t)$ of the form:

$$S(t) = \psi_0 \psi_t = \sum_n |c_n|^2 \exp\left(-i \frac{E_n t}{\hbar}\right). \tag{127}$$

Consider now the real part of $S(t)$:

$$\mathrm{Re}\big(S(t)\big) = \sum_n |c_n|^2 \cos\left(\frac{E_n t}{\hbar}\right), \tag{128}$$

which, using the trigonometric inequality:

$$\forall\, x \geq 0, \quad \cos(x) \geq 1 - \frac{2}{\pi}\big(x + \sin(x)\big), \tag{129}$$

may be rewritten in the form:

$$\sum_n |c_n|^2 \cos\left(\frac{E_n t}{\hbar}\right) \geq \sum_n |c_n|^2 \left[1 - \frac{2}{\pi}\left(\frac{E_n t}{\hbar} + \sin\left(\frac{E_n t}{\hbar}\right)\right)\right], \tag{130}$$

where:

$$\sum_n | c_n |^2 \left[1 - \frac{2}{\pi}\left(\frac{E_n t}{\hbar} + \sin\left(\frac{E_n t}{\hbar}\right)\right)\right] = \\ 1 - \frac{2}{\pi} \frac{\langle \hat{H} \rangle}{\hbar} t + \frac{2}{\pi} \mathrm{Im}\big(S(t)\big), \tag{131}$$

assuming throughout that the average energy is non-negative, that is, $\langle \hat{H} \rangle \geq 0$. If the initial and final states are orthogonal, then by definition $S(t) = 0$, and therefore also $\mathrm{Re}\big(S(t)\big) = \mathrm{Im}\big(S(t)\big) = 0$, allowing us to rearrange equation ((130)) to yield a minimum evolution time between the two orthogonal states:

$$t \geq \frac{\pi}{2} \frac{\hbar}{\langle \hat{H} \rangle}, \tag{132}$$

which is, elsewhere in the literature, referred to as the "Margolus–Levitin bound" [68]. We note with satisfaction that our aforementioned result, namely the fact that the minimum evolution time between orthonormal eigenstates (i.e., distinct global states) in the multiway graph is restricted by the maximum entanglement rate $\hbar$ in multiway space, is therefore provably consistent with the Margolus–Levitin bound.

If one attempts to approach the maximum rate of entanglement, effectively by constructing an arbitrarily steep foliation of the multiway graph, one would expect the perceived evolution of global states in the multiway system to proceed more slowly, by analogy to the phenomenon of relativistic time dilation (and, more directly, as a consequence of the geometry of multiway space). This, too, is an effect that one observes within standard quantum mechanical formalism [69]; consider now an interpretation of the quantity $S(t)$ as the "survival probability" of the initial state $| \psi_0 \rangle$ [70]:

$$S(t) = \left| \langle \psi_0 | \exp\left(-i \frac{\hat{H} t}{\hbar} \right) | \psi_0 \rangle \right|^2, \tag{133}$$

after time t, when the system has evolved to state $| \psi_t \rangle$:

$$| \psi_t \rangle = \exp\left(-i \frac{\hat{H} t}{\hbar} \right) | \psi_0 \rangle. \tag{134}$$

For sufficiently small values of t, one is able to perform a power series expansion:

$$\exp\left(-i \frac{\hat{H} t}{\hbar} \right) \approx \hat{I} - i \frac{\hat{H} t}{\hbar} - \frac{1}{2} \frac{\hat{H}^2 t^2}{\hbar^2} + \ldots, \tag{135}$$

such that the survival probability now becomes:

$$S(t) \approx \left[1 - \frac{\left(\Delta \hat{H} \right)^2 t^2}{\hbar^2} \right], \tag{136}$$

where we have introduced the quantity:

$$\left(\Delta \hat{H} \right)^2 = \langle \psi_0 | \hat{H}^2 | \psi_0 \rangle - \psi_0 \hat{H} | \psi_0 \rangle^2. \tag{137}$$

Suppose now that the system is allowed to evolve over a finite time interval $[0, t]$, but where this interval is punctuated by measurement operations (which we can idealize as being instantaneous projections) applied periodically at times t / n, $2t / n$, and so on, for some $n \in \mathbb{N}$. Assuming that the state $| \psi_0 \rangle$ is an eigenstate of the measurement operator, this implies that the survival time now takes the form:

$$S(t) \approx \left[1 - \frac{(\Delta \hat{H})^2}{\hbar^2} \left(\frac{t}{n}\right)^2\right]^n \to 1, \tag{138}$$

in the limit as $n \to \infty$. The convergence to unity of the survival probability of the initial state indicates that the evolution of the quantum system slows down as the rate of application of measurement operations increases: this phenomenon is known as the "quantum Zeno effect" and plays the role of relativistic time dilation in our discussion of multiway foliations and branchlike hypersurfaces. It is the Zeno effect that effectively prevents the maximum entanglement rate from ever being physically exceeded.

One consequence of this idea is that singularities in the multiway causal graph correspond to pure quantum eigenstates that remain isolated from the remainder of the multiway system and do not undergo evolution in time (these states, in turn, correspond to normal forms of the underlying abstract rewriting system). As described in more detail in [2], when these singularities are physical (i.e., intrinsic geometrical features of the underlying multiway space), they can be thought of as corresponding to qubits, whereas when they are purely coordinate singularities, they can be thought of as corresponding to measurement outcomes. Therefore, the act of performing a measurement consists of an observer constructing a particular multiway coordinate frame that places a coordinate singularity around a particular eigenstate (the apparent convergence of multiway geodesics due to "gravitational" lensing that results from the presence of such a coordinate singularity thus causes the resolution of multiway branch pairs, as per the Knuth–Bendix completion algorithm described earlier). This construction is analogous to the appearance of a coordinate singularity at $r = r_s = 2GM / c^2 r$ in the Schwarzschild coordinate frame (t, r, θ, ϕ):

$$g = \left(1 - \frac{2GM}{c^2 r}\right) dt^2 - \frac{dr^2}{\left(1 - \frac{2GM}{c^2 r}\right)} - r^2 d\theta^2 - r^2 \sin^2(\theta)\, d\phi^2, \tag{139}$$

which is provably not a physical feature of the underlying spacetime, since it disappears for alternative coordinate frames, such as in the Gullstrand–Painlevé coordinates (t_r, r, θ, ϕ) [71–73]:

$$g = \left(1 - \frac{2GM}{c^2 r}\right) dt_r^2 - 2\sqrt{\frac{2GM}{c^2 r}} \, dt_r dr - dr^2 - r^2 d\theta^2 - r^2 \sin^2(\theta) d\phi^2, \tag{140}$$

with the new time coordinate defined by:

$$t_r = t - a(r), \tag{141}$$

where:

$$a(r) = -\int \frac{\sqrt{\frac{2GM}{c^2 r}}}{1 - \frac{2GM}{c^2 r}} dr = 2M\left(-2y + \log\left(\frac{y+1}{y-1}\right)\right), \tag{142}$$

and:

$$y = \sqrt{\frac{c^2 r}{2GM}}\,. \tag{143}$$

Note that the maximum rate of quantum entanglement in the multiway graph, as measured by $\hbar$, is, in general, much higher than the speed of light in the purely relativistic causal graph; however, one implication of the geometry of the multiway causal graph is that this will cease to be the case in the presence of a sufficiently high density of causal edges, as is the case in the vicinity of a black hole. For this reason, we can think of a black hole in the multiway causal graph as being characterized by the presence of two distinct horizons: a "standard" event horizon, corresponding to regular causal disconnection, and an "entanglement" event horizon, corresponding to multiway disconnection, and which lies strictly on the exterior of the causal event horizon in multiway space. Therefore, from the point of view of an external observer embedded within a particular foliation of the multiway causal graph, and who is watching an infalling object to the black hole, the object will appear to "freeze" (due to quantum Zeno effects) at the entanglement horizon and will never appear to approach the true causal event horizon. Since Hawking radiation [74] (which, in this model, occurs as a consequence of branch pairs in the multiway graph that are unable to converge due to the presence of a multiway disconnection) is emitted from the entanglement horizon and not the causal event horizon, any particles that get radiated from the black hole can be perfectly correlated with the information content of the infalling object, without any apparent or actual violation of causal invariance (since no information ever crossed a spacetime/causal event horizon). The Wolfram Model therefore presents a possible resolution to the black hole information paradox that is formally rather similar to the standard resolutions implied by the holographic principle [75] and the AdS/CFT duality [76] (involving stretched horizons, complementarity, firewalls, etc. [77–80]), which we intend to explore more fully in a future publication.

Indeed, the multiway causal graph allows one to construct a toy model of holography by considering "walling off" a bundle of causal edges that corresponds to a particular branch of multiway evolution, such that all causal edges within the boundary of the wall correspond to edges in the (purely relativistic) causal graph, while all causal edges intersecting the boundary of the wall correspond to edges in the (purely quantum mechanical) multiway graph. In so doing, one obtains a duality between the bulk gravitational theory on the interior of the wall (with edges designating causal relations between events in spacetime), and the boundary quantum mechanical theory on the surface of the wall (with edges designating causal relations between events in "branchtime"), just as in AdS/CFT. One question raised by this analysis that we have yet to address, however, is the precise nature of the multiway causal graph (and its continuum limit) when considered as an abstract mathematical structure.

Our conjecture here (which we make no attempt to prove in the context of the present paper) is that the limiting structure of the multiway causal graph is, in fact, some generalization of the correspondence space of twistor theory. The standard twistor correspondence, at least in Penrose's original formulation [81, 82], designates a natural isomorphism between sheaf cohomology classes on a real hypersurface of $\mathbb{CP}^3$ (i.e., twistor space), and massless Yang–Mills fields on Minkowski space, of the form:

$$\left(\begin{array}{ccc} & \mathrm{Gr}_{1,2}\left(\mathbb{C}^4\right) & \\ \swarrow & & \searrow \\ (\mathbb{CP})^3 & & \mathrm{Gr}_2\left(\mathbb{C}^4\right) \end{array}\right) \cong \left(\begin{array}{ccc} & \mathrm{SL}_{\mathbb{C}}(4)/\mathrm{SL}_{\mathbb{C}}(2) & \\ \swarrow & & \searrow \\ \mathrm{SL}_{\mathbb{C}}(4)/\mathrm{SL}_{\mathbb{C}}(3) & & \mathrm{SL}_{\mathbb{C}}(4)/\left(\mathrm{SL}_{\mathbb{C}}(2)\times\mathrm{SL}_{\mathbb{C}}(2)\right) \end{array}\right), \tag{144}$$

where the twistor space is the Grassmannian of lines in complexified Minkowski space $\mathbb{CP}^3 = \mathrm{Gr}_1\left(\mathbb{C}^4\right)$, the massless Yang–Mills fields correspond to the Grassmannian of planes in the same space $\mathrm{Gr}_2\left(\mathbb{C}^4\right)$, and the correspondence space is therefore given by the Grassmannian of lines in planes in complexified Minkowski space $\mathrm{Gr}_{1,2}\left(\mathbb{C}^4\right)$. This correspondence space encodes both the quantum mechanical structure of the Yang–Mills fields and the geometrical structure of the background spacetime, in a manner that is directly analogous to the multiway causal graph, whose causal edges between branchlike-separated updating events encode the quantum mechanical structure of the multiway evolution, and whose causal edges between spacelike-separated updating events encode the relativistic structure of the pure (spacetime) causal graph. The natural generalization of this correspondence space would be a quotient group of the form $G / (P_1 \cap P_2)$, with a generalized twistor correspondence of the form [83]:

$$\begin{array}{ccc} & G / (P_1 \cap P_2) & \\ \swarrow & & \searrow \\ G / P_1 & & G / P_2, \end{array} \tag{145}$$

for some semisimple Lie group G and a pair of parabolic subgroups P_1, P_2.

3.4 Bell's Theorem, Particles and Consequences of Multiway Relativity

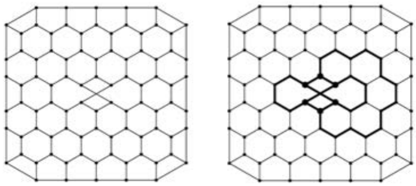
As was pointed out by Wolfram in [3], one can imagine the analog of an elementary particle in a network or hypergraph as being any persistent localized structure exhibiting certain graph-theoretic properties that get preserved by the updating rules (these properties may, in turn, be interpreted as conserved physical quantities such as electric charge). Wolfram went on to propose an illustrative example of such a conservation scheme, and therefore a toy model of how elementary particles might work in network-based spacetimes, by considering updating rules that preserve graph planarity; any local non-planar structures will then behave like persistent, particle-like excitations. The first purpose of this subsection is to present a formal version of this argument, and then to illustrate how it may be generalized to yield a combinatorial analog of Noether's theorem.

We begin by considering a fundamental result in graph theory, known as "Kuratowski's theorem" [84, 85], which states that a graph is planar (i.e., can be embedded in the plane without any crossings of edges) if and only if it does not contain a subgraph that is a subdivision of K_5 (the complete graph on five vertices) or $K_{3,3}$ (the "utility graph," or bipartite complete graph on 3 + 3 vertices).

Definition 46. A "subdivision" of an undirected graph $G = (V, E)$ is a new undirected graph $H = (W, F)$ resulting from the subdivision of edges in G.

Definition 47. A "subdivision" of an edge $e \in E$, where the endpoints of e are given by $u, v \in V$, is defined by the introduction of a new vertex $w \in W$ and the replacement of e by a new pair of edges $f_1, f_2 \in F$, whose endpoints are given by $u, w \in W$ and $w, v \in W$, respectively.

In the particular case of a trivalent spatial graph, this implies that any non-planarity in the graph must be associated with a finite set of isolable non-planar "tangles," each of which is a subdivision of $K_{3,3}$, as shown in Figure 14.



(a) (b)

Figure 14. (a) A trivalent spatial graph containing an irreducible crossing of edges. (b) An illustration of how this irreducible crossing is ultimately due to the presence of a non-planar "tangle"; this is a subdivision of $K_{3,3}$. Adapted from [3, p. 527].

However, we view this planarity example as being no more than a toy model, on the basis that Kuratowski's theorem is excessively restrictive: it only allows one to consider the existence of two possible values for locally conserved quantities in the graph, associated with the two different types of elementary particle (i.e., corresponding to K_5 and $K_{3,3}$). As a first step toward generalizing the basic idea behind Kuratowski's theorem, we next examine a stronger variant known as "Wagner's theorem" [86], which states that a graph is planar if and only if its set of minors does not contain either K_5 or $K_{3,3}$.

Definition 48. A "minor" of an undirected graph $G = (V, E)$ is a new undirected graph $H = (W, F)$ resulting from the deletion of edges from E, the deletion of vertices from V and the contraction of edges in G.

Definition 49. A "contraction" of an edge $e \in E$, where the endpoints of e are given by $u, v \in V$, is defined by the deletion of e and the merging of $u, v \in V$ into a new, single vertex $w \in W$ where the set of edges incident to w corresponds to the set of edges incident to either u or v.

This is a more general statement than that of Kuratowski, in the sense that every graph subdivision may be converted into a minor of the same type, but not every graph minor may be converted into a subdivision of the same type.

In the context of modern combinatorics, Wagner's theorem is conventionally viewed as being a special case of the vastly more general "Robertson–Seymour theorem" [87, 88], which states formally that the set of all undirected graphs, when partially ordered by the relationship of taking graph minors, forms a well-quasi-ordering. A more intuitive formulation of the same statement is that every family of graphs that is closed under the operation of taking graph minors may be uniquely characterized by some finite set of "forbidden minors," which may be thought of as being akin to a set of topological obstructions. Indeed, the Robertson–Seymour theorem can be interpreted as corresponding to a generalization of Kuratowski's embedding obstructions of K_5 and $K_{3,3}$ to higher genus surfaces (i.e., in topological terms, for every integer $n \geq 0$, there exists a finite set of graphs $\mathcal{G}(n)$, with the property that a graph G can be embedded on a surface of genus n if and only if G does not contain any of the graphs in $\mathcal{G}(n)$ as a minor). In this regard, Wagner's theorem concerns specifically the family of planar graphs (which is, trivially, closed under the operation of taking graph minors), in which case the set of forbidden minor obstructions corresponds to $\{K_5, K_{3,3}\}$.

As the Robertson–Seymour theorem allows us to define a conserved graph quantity (taking a range of possible values, each of which corresponds to a distinct forbidden minor, and hence to a distinct persistent, localized, particle-like excitation in the graph) associated to each family of graphs that is closed under the operation of taking minors, we propose that such graph families may be thought of as corresponding to differentiable symmetries of the action defined by the multiway evolution. In this way, the Robertson–Seymour theorem may be interpreted as being a combinatorial analog of Noether's theorem [89], in which differentiable symmetries of the action are bijectively associated with conservation laws.

In [3] Wolfram proposed that a pair of such particles could be connected by long-range edges in the spatial graph that, in the continuum limit, did not correspond to the usual $3+1$-dimensional spacetime structure (thus creating a rather explicit example of a quantum entanglement phenomenon). This was a welcome observation, since it allowed the model to violate the CHSH inequality [90]:

$$\left|E(a, b) - E(a, b') + E(a', b) + E(a', b')\right| \leq 2, \tag{146}$$

and therefore maintain consistency with Bell's theorem [91], without ever violating the underlying determinism of the model, since the two particles could remain causally connected in a nonlocal fashion, but without deviating from macroscopic causal invariance in the continuum limit. In equation (146), a, a' and b, b' denote the pairs of possible settings for two hypothetical detectors A and B, and the terms $E(a, b)$, $E(a, b')$, $E(a', b)$ and $E(a', b')$ denote the quantum correlations of the pairs of particles, that is, the statistical averages of $A(a) \cdot B(b)$, $A(a) \cdot B(b')$, $A(a') \cdot B(b)$ and $A(a') \cdot B(b')$, for some hypothetical measurement operation, with the discrete set of possible outcomes in both cases being $\{+1, -1\}$.

However, our new interpretation of quantum mechanics in multiway Wolfram Model systems gives us another, more subtle, means of ensuring compatibility with Bell's theorem, as a consequence of the structure of the multiway causal graph. As the multiway causal graph allows for the existence of causal connections not only between updating events on the same branch of evolution history, but also between updating events on distinct branches of evolution history, one immediately obtains an explicitly nonlocal theory of multiway evolution; more precisely, one indirectly extends the notion of causal locality beyond mere spatial locality, since pairs of events that are branchlike local will not, in general, also be spacelike local. Consequently, one is able to prove violation of the CHSH inequality in much the same way as one does for other standard deterministic and nonlocal interpretations of quantum mechanics, such as the de Broglie–Bohm or causal interpretation [92, 93].

To determine the effective dynamics of these particle-like excitations, we consider making a small perturbation to a localized region of a spatial hypergraph and then examining how that perturbation spreads through the rest of the multiway system. Recall that the sums of incoming path weights for vertices in the multiway graph, as computed using the discrete multiway norm, allow one to define discrete amplitudes for each multiway path; since each multiway edge corresponds to a (local) updating event involving a certain collection of vertices in the spatial hypergraph, it is therefore possible to associate each such spatial vertex with a scalar quantity corresponding to the sum of relevant (incoming) multiway path weights. The propagation of the initial hypergraph perturbation throughout the multiway graph is therefore described by a diffusion equation for this scalar quantity, with a discrete Laplacian operator taking the place of the regular (continuous) Laplacian [94, 95]:

Definition 50. If $G = (V, E)$ is a (hyper)graph, and $\phi : V \to R$ is a function taking values in a ring R, then the "discrete Laplacian" acting on ϕ, denoted Δ, is given by:

$$\forall\, v \in V, \quad (\Delta\phi)(v) = \sum_{w \in V : d(w,v)=1} [\phi(v) - \phi(w)], \tag{147}$$

where $d(w, v)$ denotes the natural combinatorial metric on the (hyper)graph, i.e. the graph distance between vertices w and v.

Δ is thus related to a local averaging operator M for the quantity ϕ over neighboring vertices in the hypergraph:

$$\forall\, v \in V, \quad (M\phi)(v) = \frac{1}{\deg(v)} \sum_{w : d(w,v)=1} \phi(w). \tag{148}$$

For our hypergraph perturbation case, ϕ denotes the natural "sum of path weights" quantity, with the ring R simply corresponding to the natural numbers $\mathbb{N}$.

However, we have yet to consider the (potentially constructive or destructive) effects of other hypergraph perturbations on the particular perturbation of interest; the effect of these rival perturbations may be described by an additional potential function defined over the hypergraph, denoted $P : V \to R$, or equivalently by a multiplicative operator acting diagonally on our quantity ϕ:

$$(P\phi)(v) = P(v)\phi(v). \tag{149}$$

The combined effects of all perturbations is therefore given by the operator sum:

$$H = \Delta + P, \tag{150}$$

which has the form of the discrete Schrödinger operator, that is, the discrete analog of the regular (continuous) Schrödinger operator:

$$D = -\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r}, t), \tag{151}$$

as it appears in the time-dependent Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r}, t) = \left[\frac{-\hbar^2}{2m}\nabla^2 + V(\mathbf{r}, t)\right]\psi(\mathbf{r}, t). \tag{152}$$

Our diffusion equation assumes the form of a Schrödinger equation rather than a regular heat equation because of the presence of imaginary branchlike distances in the discrete multiway metric.

By viewing our hypergraph perturbation as an elementary impulse in the quantity ϕ, we see that the response to this impulse will be given by the Green's function [96], denoted G, of the discrete Schrödinger operator H:

$$G(v, w; \lambda) = \left\langle \delta v \middle| \frac{1}{H - \lambda} \middle| \delta_w \right\rangle, \tag{153}$$

which we have here written in the resolvent formalism [97]:

$$R\left(z; A\right) = \left(A - zI\right)^{-1}, \tag{154}$$

for a given operator A, and where δ denotes a Kronecker delta function defined over the hypergraph:

$$\delta_v(w) = \delta_{vw} = \begin{cases} 1, & \text{if } v = w, \\ 0, & \text{otherwise.} \end{cases} \tag{155}$$

For a given vertex $w \in V$ and a complex number $\lambda \in \mathbb{C}$, this Green's function may be considered to be a function of v that is the unique solution to the equation:

$$(H - \lambda)G(v, w; \lambda) = \delta_w(v). \tag{156}$$

Our central conjecture here (which we intend to explore more fully in a subsequent publication) is therefore that, in the continuum limit, this discrete Green's function is related to the standard propagator from nonrelativistic quantum mechanics, that is, the kernel of the Schrödinger operator that corresponds physically to the transition amplitude between state x at time t and state x' at time t' [98]:

$$K(x, t; x', t') = \langle x|\hat{U}(t, t')|x'\rangle, \tag{157}$$

since the discrete Green's function will, in this limit, converge to the standard Green's function for the time-dependent Schrödinger equation:

$$G(x, t; x', t') = \frac{1}{i\hbar}\Theta(t - t')K(x, t; x', t'), \tag{158}$$

that is itself the solution to the equation:

$$\left(i\hbar\frac{\partial}{\partial t} - \hat{H}_x\right)G(x, t; x', t') = \delta(x - x')\delta(t - t'), \tag{159}$$

where $\hat{H}_x$ denotes a Hamiltonian operator in x position coordinates, $\delta(x)$ is the Dirac delta function, $\Theta(t)$ is the Heaviside step function, and $\hat{U}(t, t')$ is the unitary time evolution operator for the system between times t and t'.

It is worth noting that this propagator is nonrelativistic as a consequence of our choice to perform this analysis with respect to the multiway evolution graph (which is purely quantum mechanical) as opposed to the full multiway causal graph (which also incorporates relativistic effects). The extension of these methods to the multiway causal graph, and therefore a more complete derivation of the mathematical apparatus of quantum field theory, will be outlined in a forthcoming paper.

4. Concluding Remarks

The present paper has indicated the potential relevance of the Wolfram Model in addressing a variety of open questions related to the foundations of quantum mechanics and has presented possible approaches for deriving many further aspects of the mathematical formalism of quantum information theory and quantum field theory from discrete hypergraph dynamics. Many directions for future research arise out of the present paper, ranging from a more systematic investigation of the physical implications of multiway relativity to more rigorous mathematical exploration of the limiting structures and geometry of the multiway evolution and causal graphs, to greater elucidation of the implications of the Wolfram Model for holography, black hole information and the AdS/CFT correspondence. We intend to address several of these questions (particularly with regard to computability-theoretic and information-theoretic aspects of the formalism) in future publications, to be released throughout the remaining course of the Wolfram Physics Project, but our hope is also that these directions may sow the seeds for novel and exciting research programs in their own right.

Acknowledgments

The author is greatly indebted to the other members of the Wolfram Physics Project, namely Stephen Wolfram and Max Piskunov, for their many and various contributions and insights related to the present work. Stephen is responsible for originally proposing the formalism of the Wolfram Model, for determining many of its salient empirical and conceptual properties, for contributing greatly to the development of the intellectual framework of which this work is a part, and (most notably through the writing of [3]) for inventing and continuing to champion the very scientific paradigm in which this work was conducted. Max is responsible for much of the software development that has spearheaded the present project, and in particular remains the principal developer of the *SetReplace* paclet [99], used (among other things) to render the images in Section 2.1 of the present paper.

The author would also like to thank Isabella Retter and Ed Pegg Jr for useful conversations (regarding discrete differential geometry and combinatorial enumeration, respectively), Øyvind Tafjord and Todd Rowland for their ample supply of NKS-related source material and Sushma Kini for keeping us all on speaking terms.

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