Evolutions of Some One-Dimensional Homogeneous Cellular Automata

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Evolution patterns of a one-dimensional homogeneous cellular automaton (CA) are investigated for some standard transition functions. The different possible evolution patterns for an elementary CA starting with at most one active cell or ON state cell are discussed. Also, with respect to some initial configurations, evolution-wise equivalent Wolfram codes are investigated. It is shown that these equivalent codes are automorphic.

Keywords: cellular automaton; evolution pattern; evolution-wise equivalent Wolfram code; elementary cellular automaton

1. Introduction

A cellular automaton (CA) is a computational model of a dynamical system. This model was introduced by J. von Neumann and S. Ulam in 1940 for designing self-replicating systems that later saw applications in physics, biology and computer science.

Neumann conceived a CA as a two-dimensional mesh of finite-state machines called cells that are locally interconnected with each other. Each of the cells changes its states synchronously, depending on the states of some neighboring cells (for details see [1–3] and references therein). The local changes of each of the cells together induce a change of the entire mesh. Later, a one-dimensional CA, that is, a CA where the elementary cells are distributed on a straight line, was studied. Stephen Wolfram's work in the 1980s contributed to a systematic study of one-dimensional cellular automata (CAs), providing the first qualitative classification of their behavior (reported in [4, 5]).

This paper considers only synchronous (all the cell states are updated simultaneously) CAs where the underlying topology is a one-dimensional grid line. A finite-state semi-automaton with finite memory models a simple computation. A CA is a computational model of a dynamical system where a finite/countably infinite number of semi-automata (*cells*) are arranged in an ordered linear grid [6, 7] (Figure 1).

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…	…	A_{i-1}	A_i	A_{i+1}	…	…

Figure 1. A typical grid of a linear CA.

Each cell works synchronously, leading to evolution of the entire grid through a number of discrete time steps. If the set of memory elements of each semi-automaton is $\{0, 1\}$, then a typical pattern evolved over time t may be as shown in Figure 2.

Time ↓	Grid Position (i) →	…	–3	–2	–1	0	1	2	3	…
$t = 0$	Configuration C^0 →	…	0	0	0	1	0	0	0	…
$t = 1$	C^1 →	…	0	0	1	0	1	0	0	…
$t = 2$	C^2 →	…	0	1	0	1	0	1	0	…
$t = 3$	C^3 →	…	1	0	1	0	1	0	1	…
⋮	⋮	…	⋮	⋮	⋮	⋮	⋮	⋮	⋮	…

Figure 2. The grid line at time t gives the configuration of the CA at time t.

Algebraic properties of a CA and its relation to group theory and topology have been the focus of increasing study in recent years (see [8–12]). The study of invertible CAs has also been of special interest [13–15]. All the CAs referred to are deterministic in nature. Nondeterministic CAs have also been studied in [16].

In this paper, one-dimensional synchronous homogeneous CAs are studied. In Section 2, basic concepts are introduced and some fundamental results are reported. Evolution patterns of homogeneous CAs under some standard transition functions are studied in Section 3. Section 4 is devoted to elementary CAs (ECAs) starting with at most one cell in the active or ON state. Also, some evolution-wise equivalent Wolfram codes for such ECAs are discussed. It could be shown that there exists an automorphism between transition functions represented by these Wolfram codes.

2. Basic Concepts

We give a formal definition of a CA.

Definition 1. Let us consider a finite set Q called the state set. The memory elements of the automata placed on the grid line belong to this state set Q.

A global configuration is a mapping from the group of integers $\mathbb{Z}$ to the set Q given by $C: \mathbb{Z} \to Q$.

The set $Q^{\mathbb{Z}}$ is the set of all global configurations where $Q^{\mathbb{Z}} = \{C \mid C: \mathbb{Z} \to Q\}$.

A mapping $\tau : Q^{\mathbb{Z}} \to Q^{\mathbb{Z}}$ is called a global transition function.

A CA (denoted by C_τ^Q) is a triplet $\left(Q, Q^{\mathbb{Z}}, \tau\right)$ (see [17]) where Q is the finite state set, $Q^{\mathbb{Z}}$ is the set of all configurations and τ is the global transition function.

Remark 1. For a particular state set Q and a particular global transition function τ, a triple $\left(Q, Q^{\mathbb{Z}}, \tau\right)$ denoted by C_τ^Q defines the set of all possible cellular automata on (Q, τ). However, the evolution of a CA at times is dependent on the initial configuration (starting configuration) of the CA. A particular CA $C_\tau^Q\left(C^0\right) \in C_\tau^Q$ is defined as the quadruple $\left(Q, Q^{\mathbb{Z}}, \tau, C^0\right)$ such that $C^0 \in Q^{\mathbb{Z}}$ is the initial configuration of the particular CA $C_\tau^Q\left(C^0\right)$.

The configuration at time t is denoted by C^t such that $C^t \in Q^{\mathbb{Z}}$ for all time t.

Also,

$$\tau\left(C_t\right) = C^{t+1}.$$

With reference to Figure 2 we get

$$C^0 = \ldots 0\,001\,000 \ldots;$$
$$\tau\left(C^0\right) = \tau\left(\ldots 0\,001\,000 \ldots\right) = \ldots 0\,010\,100 \ldots = C^1;$$
$$\tau\left(C^1\right) = \tau\left(\ldots 0\,010\,100 \ldots\right) = \ldots 0\,101\,010 \ldots = C^2, \text{ etc.}$$

Evolution of a CA is mathematically expressed by the global transition function. However, this global transition is induced by transitions of the cells at each grid point of the CA. The transition of the state of the i^{th} cell at a particular time depends on the state of the cell at the i^{th} grid point and its adjacent cells, which constitute the neighborhood of the i^{th} cell. The transition of the cell at each grid point is called local transition.

Definition 2. For $i \in \mathbb{Z}, r \in \mathbb{N}$, let $S_i = \{i - r, \ldots, i - 1, i, i + 1, \ldots, i + r\} \subseteq \mathbb{Z}$. S_i is the neighborhood of the i^{th} cell. r is the radius of the neighborhood of a cell. It follows that $\mathbb{Z} = \bigcup_i S_i$.

A restriction from $\mathbb{Z}$ to S_i induces the following:

1. Restriction of C to $\overline{c_i}$ is given by $\overline{c_i} : S_i \to Q$, and $\overline{c_i}$ may be called a local configuration of the i^{th} cell.
2. Restriction of $Q^{\mathbb{Z}}$ to Q^{S_i} is given by $Q^{S_i} = \{\overline{c_i} \mid \overline{c_i} : S_i \to Q\}$, and Q^{S_i} may be called the set of all local configurations of the i^{th} cell.

The mapping $\mu_i : Q^{S_i} \to Q$ is known as a local transition function for the i^{th} cell having radius r. So $\forall\, i \in \mathbb{Z}, \mu_i(\overline{c_i}) \in Q$. If the local configuration of the i^{th} cell at time t is denoted by c_i^t, then $\mu_i\left(c_i^t\right) = c_i^{t+1}(i)$.

Remark 2. If $\tau(C) = C^*$, then $\forall\ i \in \mathbb{Z}$, $C^*(i) = \tau(C)(i) = \mu_i(\overline{c_i})$. So we have

1. $C^{t+1}(i) = \tau(C^t)(i) = \mu_i(c_i^t) = c_i^{t+1}(i)$.
2. $\tau(C) = \ldots\mu_{i-1}(c_{i-1}) \cdot \mu_i(\overline{c_i}) \cdot \mu_{i+1}(c_{i+1})\ldots$.

Definition 3. If all μ_i are identical, then the CA is homogeneous. A homogeneous CA may also be defined as a triplet (Q, r, μ) where Q is the finite-state set, r is the radius of the neighborhood of a cell and μ is the local transition function.

However, if all μ_i are not identical, then the CA is hybrid.

Henceforth, in this paper, CA will refer to a one-dimensional homogeneous synchronous CA.

Remark 3. The set $\boldsymbol{Q}^{\mathbb{Z}}$, where $\boldsymbol{Q}^{\mathbb{Z}} = (Q^{\mathbb{Z}})^{Q^{\mathbb{Z}}}$ is the set of all global transition functions of a CA defined as $\boldsymbol{Q}^{\mathbb{Z}} = (Q^{\mathbb{Z}})^{Q^{\mathbb{Z}}} = \{\tau \mid \tau : Q^{\mathbb{Z}} \to Q^{\mathbb{Z}}\}$.

Remark 4. The set M, where $M = \left\{(Q^{S_i})^Q \,\middle|\, i \in \mathbb{Z}\right\}$ is the set of all local transition functions of a CA defined as

$$M = \left\{(Q^{S_i})^Q \,\middle|\, i \in \mathbb{Z}\right\} = \{\mu \mid \mu : Q^{S_i} \to Q, i \in \mathbb{Z}\}.$$

If there is no ambiguity regarding Q, C^0 a CA is often denoted by τ where $\tau \in \boldsymbol{Q}^{\mathbb{Z}}$.

Definition 4. If for a particular CA, $|Q| = 2$ so that we can write $Q = \{0, 1\}$, then the CA is said to be a binary CA or a Boolean CA.

An elementary CA is a one-dimensional Boolean CA with 1-radius of neighborhood for any cell.

For a binary CA $(Q, Q^{\mathbb{Z}}, \tau)$, if $C^1, C^2 \in Q^{\mathbb{Z}}$ such that $\tau(C^1) = C^2$ where

$$\forall\ i \in \mathbb{Z},\ C^1(i) = 0 \leftrightarrow C^2(i) = 1 \text{ and } C^1(i) = 1 \leftrightarrow C^2(i) = 0,$$

then C^1 is the complement of C^2 and vice versa. τ is said to be the complement transition function and is denoted by τ^c.

Definition 5. A global transition function τ is an identity function denoted by τ_e provided for all $C \in Q^{\mathbb{Z}}$, $\tau_e(C) = C$.

A CA is said to be an identity CA if the global transition function is τ_e.

Definition 6. A global transition function τ is an m-place left shift function denoted by τ_{Lm}, where $m \in \mathbb{N}$ is finite, if $\forall\ i \in \mathbb{Z}$, $\tau_{Lm}(C)(i) = C(i + m)$.

A CA is an m-place left shift CA if the global transition function is $\tau_{\mathrm{L}m}$.

Definition 7. A global transition function τ is an m-place right shift function denoted by $\tau_{\mathrm{R}m}$, where $m \in \mathbb{N}$ is finite, if $\forall\, i \in \mathbb{Z}$, $\tau_{\mathrm{R}m}(C)(i) = C(i-m)$.

A CA is an m-place right shift CA if the global transition function is $\tau_{\mathrm{R}m}$.

Definition 8. A global transition function τ is a constant function provided for all $C \in Q^{\mathbb{Z}}$, $\tau(C) = C^*$ for a particular constant configuration $C^* \in Q^{\mathbb{Z}}$.

If $\forall\ i \in \mathbb{Z}$, $C^*(i) = q$ for some $q \in Q$, then the global transition function is denoted by τ_q.

A CA is said to be a constant CA if the global transition function is a constant function.

2.1 Binary Operations on One-Dimensional Cellular Automata with the Same State Set

Any element of the set $\boldsymbol{Q}^{\mathbb{Z}} = \left(Q^{\mathbb{Z}}\right)^{Q^{\mathbb{Z}}}$ is a global transition function, often called a CA with the state set Q.

Definition 9. A binary operation $*$ on $\boldsymbol{Q}^{\mathbb{Z}}$ [8, 11, 13] (for composition of global transition functions) is defined as follows:

If τ_1 and $\tau_2 \in \boldsymbol{Q}^{\mathbb{Z}}$, then

$$\forall\ C \in Q^{\mathbb{Z}},\ (\tau_1 * \tau_2)(C) = \tau_1(\tau_2(C)).$$

Remark 5. For a particular state set Q, the binary operation $*$ on $\boldsymbol{Q}^{\mathbb{Z}}$ is closed and associative. Moreover, since τ_e belongs to $\boldsymbol{Q}^{\mathbb{Z}}$, the set $(\boldsymbol{Q}^{\mathbb{Z}}, *)$ forms a monoid (see [11, 13]).

Remark 6. The set $(G,\ *)$ forms a group where $G = \{\tau \mid \tau \text{ is invertible}\}$ (see [13]).

Proposition 1. Any global transition function τ commutes with a shift transition function (see [11, 13]). Thus for some finite $m \in \mathbb{N}$ and $\forall\ C \in Q^{\mathbb{Z}}$,

$$(\tau * \tau_{\mathrm{L}m})(C) = (\tau_{\mathrm{L}m} * \tau)(C) \text{ and } (\tau * \tau_{\mathrm{R}m})(C) = (\tau_{\mathrm{R}m} * \tau)(C).$$

2.2 Wolfram Code

Wolfram code is a naming system often used for one-dimensional Boolean CAs, introduced by Stephen Wolfram (see [5]).

Corresponding to a particular local rule of a Boolean CA, it is a number in the range from 0 to $\left(2^{2^{2r+1}}-1\right)$ where r is the radius of neighborhood of a cell.

For example: Let the local rule μ for some i^{th} cell, $i \in \mathbb{Z}$ of radius 1 be

$$\mu(\overline{c_i}) = \mu\left(\overline{c_i}(i-1), \overline{c_i}(i), \overline{c_i}(i+1)\right) = \left(\overline{c_i}(i-1) \vee \overline{c_i}(i+1)\right) \wedge \overline{c_i}(i),$$

where $\vee$ stands for the OR operation, and $\wedge$ stands for the AND operation. Thus,

$$\mu(111)\,\mu(110)\,\mu(101)\,\mu(100)\,\mu(011)\,\mu(010)\,\mu(001)\,\mu(000) = 1\,1\,0\,0\,1\,0\,0\,0.$$

11 001 000 is the binary equivalent of 200. So the Wolfram code is rule 200.

For an elementary CA, some special transition functions represented by the Wolfram codes are as follows:

1. Rule 204 is an identity function.
2. Rule 0 or rule 255 is a constant function.
3. Rule 51 is a complement function.
4. Rule 170 is a 1-place left shift function.
5. Rule 240 is a 1-place right shift function.

3. Patterns of Evolution of a Homogeneous Cellular Automaton under Some Standard Transitions

The evolution pattern of a homogeneous CA for identity, constant, complement and shift transition functions is discussed in this section.

Proposition 2. A homogeneous CA with a countably infinite number of cells having the global transition function:

1. τ_e is stable from the initial step.
2. τ^c is oscillatory, having 2-period cycles from the initial step.
3. τ_q is stable from the initial step or after the first transition step for some $q \in Q$.

Proof.

1. From definition of τ_e, the result follows.
2. $\forall\ C \in Q^{\mathbb{Z}}$, $\tau^c * \tau^c(C) = (\tau^c)^2(C) = C$.
3. Let $C \in Q^{\mathbb{Z}}$ be such that $C(i) = q \forall\ i \in \mathbb{Z}$.

Then $\tau_q(C) = C$ and the CA is stable from the initial step.

Again let $C \in Q^{\mathbb{Z}}$ be such that $C(i) \neq q$ at least for some $i \in \mathbb{Z}$.

Then $\tau_q(C) = C^*$ where $C^*(i) = q \forall\ i \in \mathbb{Z}$ and the CA becomes stable after the first step.

Hence the proposition. □

Remark 7. The evolution pattern of a homogeneous CA with a finite number of cells under τ_e, τ^c and τ_q is similar to that of a homogeneous CA with a countably infinite number of cells.

Proposition 3. In a binary CA, let τ^c be a complement global transition function and τ_q be a constant global transition function for $q \in \{0, 1\}$ such that q^* is the complement of q. Then $\forall\ C \in Q^{\mathbb{Z}}$, if $\overline{C}$ is the complement of the configuration C,

$$(\tau_q * \tau^c)(C) = \overline{(\tau^c * \tau_q)(C)},$$

where $\overline{(\tau^c * \tau_q)(C)}$ is the complement of $(\tau^c * \tau_q)(C)$.

Proof. Let $C \in Q^{\mathbb{Z}}$. Then $\forall\ i \in \mathbb{Z}$,

$$(\tau_q * \tau^c)(C(i)) = \tau_q(\tau^c(C(i))) = q.$$

Again,

$$\overline{(\tau^c * \tau_q)(C)}(i) = \tau^c(\tau_q(C(i)))^* = (\tau^c(q))^* = (q^*)^* = q.$$

Hence the result. □

Proposition 4. Let τ_q be a constant global transition function for some $q \in Q$. Let τ be any global transition function having $\mu(\overline{c_i}) = q$.

If $\forall\ j \in S_i \subseteq \mathbb{Z}$, $\overline{c_i}(j) = q$, then τ_q commutes with τ.

Proof. Let τ_q be a constant global transition function for some $q \in Q$. Then $\forall\ C \in Q^{\mathbb{Z}}$, $\forall\ i \in \mathbb{Z}$,

$$\tau_q(C)(i) = q.$$

Let $\overline{c_i} \in S_i \subseteq \mathbb{Z}$ be such that $\forall\ j \in S_i$, $\overline{c_i}(j) = q$.

If τ is any global transition function having local transition $\mu(\overline{c_i}) = q$, then

$$\forall\ j \in S_i \subseteq \mathbb{Z},\ \tau(C)(j) = q.$$

Since $\mathbb{Z} = \bigcup_i S_i$, it follows that $\forall\ i \in \mathbb{Z}$,

$$\tau * \tau_q(C)(i) = q = \tau_q * \tau(C)(i).\ \square$$

Proposition 5. For a homogeneous CA with a countably infinite number of cells under transition function $\tau_{\mathrm{L}m}$ for some finite $m \in \mathbb{N}$ having an initial configuration that is not block repetitive, the evolution pattern will yield countably infinite global configurations.

Proof. The cell states shift m places leftward at each discrete time step under transition function $\tau_{\mathrm{L}m}$. For a CA with a countably infinite number of cells, this transition pattern continues for all time, giving countably infinite global configurations, since the initial configuration is not block repetitive. □

Proposition 6. For a homogeneous CA with n cells with periodic boundaries evolving under transition function $\tau_{\mathrm{L}m}$ for some finite $m \in \mathbb{N}$,

$$s = \frac{n}{\gcd(m, n)}$$

if $m < n \in \mathbb{N}$.

Proof. Let a CA with n cells with periodic boundaries have the transition function $\tau_{\mathrm{L}m}$ where $m < n$. As the transition pattern continues, the cell states shift m places leftward at each discrete time step. So, $\forall\, i \in \{1, 2, \ldots n\}$,

$$\tau\big(C(i)\big) = C(i + m).$$

Let s be the smallest integer such that

$$\tau^s\big(C(i)\big) = C(i + sm) = C(i).$$

As n is finite, it will always be possible to find one such s. Since the initial configuration is not block repetitive, the initial configuration reappears at every $s = n \big/ \gcd(m, n)$ step. Hence the result follows. □

Proposition 7. A homogeneous CA with periodic boundaries and a countably infinite number of cells or n cells under transition function $\tau_{\mathrm{L}m}$ having a repetitive block configuration of k cells for some finite $m, k \in \mathbb{N}$ is

1. s-periodic where
   - $s = k \,/\, \gcd(m, k)$ if $m < k < n$
   - $s = k \,/\, \gcd(m - k, k)$ if $k < m < n$
2. stationary from initial step if $m = k < n$

Proof. A homogeneous CA with a repetitive block configuration of k cells for some finite $k \in \mathbb{N}$ can be treated as a CA of k cells with periodic boundaries having an initial configuration that is not block repetitive.

1. Thus under $\tau_{\mathrm{L}m}$, the initial configuration reappears at every
   - $s = k / \gcd(m, k)$ step if $m < k < n$
   - $s = k / \gcd(m - k, k)$ step if $k < m < n$.
2. Again, if $m = k < n$, then since the cell states shift k places leftward at every discrete time step, the initial configuration always remains stationary.

Hence the result follows. □

Remark 8. A homogeneous CA with periodic boundaries and n cells under transition function τ_{Ln} for some finite $n \in \mathbb{N}$ is stationary from the initial step, since the cell states shift n places leftward at every discrete time step, keeping the initial configuration stationary.

Proposition 8. A homogeneous CA with n cells with boundaries fixed to $q_B \in Q$ having transition function $\tau_{\mathrm{L}m}$ for some finite $m \leq n \in \mathbb{N}$ becomes stationary with all cells acquiring the q_B state after at most s steps where

$$s = \mathrm{ceil}\left(\frac{n}{m}\right) = \left\lceil \frac{n}{m} \right\rceil.$$

Proof. The cell states shift m places leftward at each discrete time step under transition function $\tau_{\mathrm{L}m}$ and m rightmost cells of the CA acquire the q_B state, the state of the right boundary.

If $\exists\ j < n \in \mathbb{N}$ such that $\overline{c_i}(i) = q_B\ \forall\ i = j + 1, \ldots, n$, then this process continues and all the cells eventually acquire the q_B state after s steps where

$$s = \mathrm{ceil}\left(\frac{j}{m}\right) = \left\lceil \frac{j}{m} \right\rceil.$$

However, if $\nexists\ j < n \in \mathbb{N}$ such that $\overline{c_i}(i) = q_B\ \forall\ i = j + 1, \ldots, n$, then all the cells acquire the q_B state after $s = \left\lceil \frac{n}{m} \right\rceil$ steps.

Hence the proposition. □

Remark 9. Similarly, evolution patterns of a homogeneous CA under transition function $\tau_{\mathrm{R}m}$ for some finite $m \in \mathbb{N}$ can be obtained where the cell states shift m places rightward at each discrete time step.

4. Elementary Cellular Automata Having at Most One Cell in the ON State Initially

Evolution patterns of some ECAs with a countably infinite number of cells starting with at most one cell in the ON state are studied in this section. Thus, an ECA considered here can either have all OFF state cells or have exactly one ON state cell initially. Let a cell having state "1" correspond to the ON state and a cell having "0" correspond to the OFF state. Clearly, the evolution of an ECA is reflected by its transition functions (represented by Wolfram codes) and hence its local configurations.

Now, two CAs are equivalent if and only if their local structures are permutations of each other [18]. Automorphism between two CAs using permutations has been discussed in [19]. Here, transition functions of some ECAs having evolution-wise equivalent Wolfram codes as defined in Definition 10 have been discussed. Also, it has been shown that these transition functions represented by evolution-wise equivalent Wolfram codes induce an automorphism.

Definition 10. Any two transition functions τ_1, τ_2, represented by Wolfram codes $w1, w2$, respectively, are evolution-wise equivalent with respect to some initial configuration $C^0 \in Q^{\mathbb{Z}}$, if $\forall\ s \in \mathbb{N}$,

$$\tau_1^s(C^0) = \tau_2^s(C^0)$$

where $\tau_i^s = \underbrace{\tau_i * \cdots * \tau_i}_{s \text{ times}}$ for $i = 1, 2$ and "$*$" represents the composition of the transition functions.

A set of evolution-wise equivalent transition functions forms a semigroup with regard to "$*$", since composition of functions is associative. Moreover, the semigroup becomes a group if the transition functions are invertible.

Theorem 1. Let $(Q_E, *)$ be a semigroup of all evolution-wise equivalent transition functions. For an initial configuration $C^0 \in Q^{\mathbb{Z}}$ and $\forall\ s \in \mathbb{N}$, a function $\rho : Q_E \to Q_E$ defined by

$$\rho(\tau_1^s(C^0)) = \tau_2^s(C^0)$$

is an automorphism.

Proof. Let $\tau_1, \tau_2 \in (Q_E, *)$. Then $\forall\ s \in \mathbb{N}$, $\tau_1^s, \tau_2^s \in Q_E$.

Now, for any $i, j \in \mathbb{N}$, it follows that

$$\begin{aligned}
\rho((\tau_1^j * \tau_1^k)(C^0)) &= \rho(\tau_1^{j+k}(C^0)) \\
&= \tau_2^{j+k}(C^0) \\
&= \tau_2^j(\tau_2^k(C^0)) \\
&= (\tau_2^j * \tau_2^k)(C^0) \\
&= \tau_2^j(C^0) * \tau_2^k(C^0) \\
&= \rho(\tau_1^j(C^0)) * \rho(\tau_1^k(C^0)).
\end{aligned}$$

Hence ρ is an automorphism. □

4.1 Evolution of Elementary Cellular Automata with a Countably Infinite Number of Cells Starting with All OFF State Cells

An ECA can have $2^3 = 8$ local configurations. The evolution pattern of an ECA is dependent on the transitions of the state of the middle cell of its local configurations.

Theorem 2. The transition function of an ECA with a countably infinite number of cells having all OFF state cells initially is either constant or complimentary.

Proof. The initial global transition of an ECA starting with all OFF state cells is dependent on the transitions of the local configuration 000. The following cases arise:

- Case 1: The middle cell of the local configuration 000 remains at state 0.

  Since all the cells of the ECA remain in the 0 state, the only possible transition function is such that $\forall\, i \in \mathbb{Z}$, $\forall\, C \in Q^{\mathbb{Z}}$, $\tau(C)(i) = 0$.

  The transition function is a constant function and can be represented by rule 0 of Wolfram as shown in Figure 3(a).

  A transition function of this ECA will be independent of the rest of the seven local configurations and thus can be represented by $2^7 = 128$ evolution-wise equivalent Wolfram codes.

- Case 2: The middle cell of the local configuration 000 changes to state 1.

  Here all the cells of the ECA change to state 1, such that $\forall\, i \in \mathbb{Z}$, $\forall\, C \in Q^{\mathbb{Z}}$, $\tau(C)(i) = 1$.

  After the first step, the evolution pattern of the ECA will depend on the transitions of the local configuration 111. The following cases arise:

  (a) If the middle cell of the local configuration 111 remains at state 1, the transition function is a constant function and can be represented by rule 255 of Wolfram as shown in Figure 3(b).

  (b) If the middle cell of the local configuration 111 changes to state 0, then the ECA will have a 2-period oscillation. The transition function will be a complimentary function and can be represented by rule 51 of Wolfram as shown in Figure 3(c).

  In either case, since the global configurations will be independent of the local configurations 001, 010, 011, 100, 101 and 110, a transition function can be represented by $2^6 = 64$ evolution-wise equivalent Wolfram codes.

Hence the theorem. □

Corollary 1. The set of all Wolfram codes for the transition function of an ECA starting with all OFF state cells can be partitioned into three equivalence classes. The three classes comprise the transition functions represented by Wolfram codes that are evolution-wise equivalent to rule 0, rule 255 or rule 51.

4.2 Evolution of Elementary Cellular Automata with a Countably Infinite Number of Cells Starting with Exactly One ON State Cell

An ECA with a countably infinite number of cells starting with exactly one ON state cell can henceforth have configurations with all OFF state cells or exactly one ON state cell or more than one ON state cell. Some cases are discussed here.

4.2.1 Elementary Cellular Automata Having at Most One Cell Always in the ON State

Theorem 3. The transition function of an ECA with a countably infinite number of cells starting with one ON state cell and always having at most one ON state cell is constant, identity or a 1-place shift function.

Proof. Let only the i^{th} cell of the ECA be ON initially.

If that cell remains ON always and the other cells are OFF, then the transition function is an identity such that $\forall\, i \in \mathbb{Z}$, $\forall\, C \in Q^{\mathbb{Z}}$,

$$\tau\big(C\big)(i) = C(i).$$

If the i^{th} cell becomes OFF, then the following cases arise:

1. All the cells become OFF in the next step.

   The transition function is a constant function (Figure 4(a)) such that $\forall\, i \in \mathbb{Z}$, $\forall\, C \in Q^{\mathbb{Z}}$, $\tau(C)(i) = 0$.

2. Since the cells of an ECA have 1 radius of neighborhood, the $(i+1)^{\text{th}}$ or $(i-1)^{\text{th}}$ cell becomes ON in the next step.

   Thus the transition is a 1-place shift function such that $\forall\, C \in Q^{\mathbb{Z}}$, $\tau(C)(i) = C(i+1)$ or $C(i-1)$.

Since the ECA is homogeneous, the theorem follows. □

4.2.2 Wolfram Codes for Elementary Cellular Automata Having Exactly One Cell Always in the ON State

As per Theorem 3, it is obvious that possible transition functions of an ECA having exactly one ON state cell are identity function, 1-place left shift function and 1-place right shift function.

Theorem 4. For an ECA with a countably infinite number of cells having exactly one cell always in the ON state, a transition function can be represented by 16 equivalent Wolfram codes.

Proof. A binary CA with 1 radius of neighborhood of cells can have 2^3 local configurations. An ECA having a global configuration with exactly one ON state cell is dependent on the local configurations 001, 010, 100 and 000 only.

The middle cell state of one of the configurations 001, 010 and 100 should transition to state 1 and the middle cell state of 000 must remain at 0. Moreover, a transition function of the CA will be independent of the rest of the local configurations, namely 011, 101, 110 and 111. Hence there can be 2^4 different combinations of binary sequences, giving 16 equivalent Wolfram codes. □

Theorem 5. The equivalent Wolfram codes of transition functions for a countably infinite ECA having exactly one cell always in the ON state are as follows:

1. Identity transition functions as shown in Figure 4(b) can be represented by rules 4, 12, 36, 44, 68, 76, 100, 108, 132, 140, 164, 172, 196, 204, 228 and 236.
2. 1-place left shift transition functions as shown in Figure 4(c) can be represented by rules 2, 10, 34, 42, 66, 74, 98, 106, 130, 138, 162, 170, 194, 202, 226 and 234.
3. 1-place right shift transition functions as shown in Figure 4(d) can be represented by rules 16, 24, 48, 56, 80, 88, 112, 120, 144, 152, 176, 184, 208, 216, 240 and 248.

Proof. It is obvious that rule 4 represents an identity transition function for an ECA. Another possible rule for an identity transition function may be $(010 \to 1, 011 \to 1)$. So the rule may be calculated as:

$$\mu(111)\mu(110)\mu(101)\mu(100)\mu(011)\mu(010)\mu(001)\mu(000) = 00\,001\,100 = 1\times 2^3 + 1\times 2^2 = 12.$$

Similarly, we may get other results. □

Corollary 2. The set of Wolfram codes for the transition function of an ECA with a countably infinite number of cells having exactly one cell in the ON state always are

$$S = \{2, 4, 10, 12, 16, 24, 34, 36, 42, 44, 48, 56, 66, 68, 74, 76, 80, 88, 98, 100, 106, 108, 112, 120, 130, 132, 138, 140, 144, 152, 162, 164, 170, 172, 176, 184, 194, 196, 202, 204, 208, 216, 226, 228, 234, 236, 240, 248\}.$$

If W is the set of Wolfram codes for the transition functions of an ECA and $w \in (W - S)$, then w cannot be the Wolfram code for the transition function of an ECA with a countably infinite number of cells having exactly one cell always in the ON state.

4.2.3 Wolfram Codes for Elementary Cellular Automata Never Having Two Adjacent Cells in the ON State

An ECA never having a global configuration with two adjacent ON state cells can be dependent on the local configurations 000, 001, 010, 100 and 101 only. Clearly, the middle cell state of the local configuration 000 must always remain at 0. Now if the middle cell state of only one of 001, 010 or 100 is at state 1, then the ECA will always have exactly one ON state cell, as discussed in Section 4.2.2. Again, local configurations 001 and 010, 100 and 010, or 001, 010 and 100 cannot be at state 1 simultaneously, since for these cases, two adjacent cells will acquire ON states. Thus the middle cell state of the local configuration 010 must remain at 0. Here, the middle cell states of the local configurations 001 and 100 have been considered to be at state 1, since 001 and 100 at state 0 would represent an ECA having all OFF state cells after the first step.

Theorem 6. For an ECA with a countably infinite number of cells never having two adjacent ON state cells, a transition function can be represented by eight evolution-wise equivalent Wolfram codes.

Proof. An ECA never having a global configuration with two adjacent ON state cells will depend on the transition of the local configurations 101. Moreover, a transition function of the ECA will be independent of the local configurations 011, 110 and 111. Hence there can be 2^3 different combinations of binary sequences, giving eight evolution-wise equivalent Wolfram codes.

The following cases can arise:

- Case 1: The middle cell of the local configuration 101 is at state 0.

  The transition function of the ECA can be represented by rule 90 of Wolfram, as shown in Figure 5(a).

  Thus the Wolfram codes that are evolution-wise equivalent to rule 90 are rules 18, 26, 82, 146, 154, 210 and 218.

- Case 2: The middle cell of the local configuration 101 is at state 1.

  The transition function of the ECA can be represented by rule 50 of Wolfram, as shown in Figure 5(b).

  Thus the Wolfram codes that are evolution-wise equivalent to rule 50 are rules 58, 114, 122, 178, 186, 242 and 250.

Hence the theorem. □

4.2.4 Wolfram Codes for Elementary Cellular Automata Having Shift Transitions

For an ECA starting with exactly one ON state cell and having shift transitions, the middle cell state of the local configuration 000 must always remain at 0. Now, if the middle cell state of the local configuration 010 changes to state 0, then the ECA will have:

- strictly 1-place shift transitions as discussed in Section 4.2.2, when either 001 or 100 changes to state 1
- transitions as discussed in Section 4.2.3, when both 001 & 100 change to state 1
- constant transition with all OFF state cells after the first step

Thus, the middle cell state of the local configuration 010 has been considered to be at state 1.

Theorem 7. For an ECA with a countably infinite number of cells starting with exactly one ON state cell, if the middle cell of the local configuration 010 remains at state 1, then a shift transition function can be represented by four evolution-wise equivalent Wolfram codes.

Proof. An ECA starting with exactly one ON state cell and having the middle cell of the local configuration 010 at state 1 will depend on the transition of the local configurations 001, 011, 100 and 110. Moreover, a transition function of the ECA will be independent of the local configurations 101 and 111. Hence there can be 2^2 different combinations of binary sequences, giving four evolution-wise equivalent Wolfram codes.

The following cases can arise:

- Case 1: The middle cell of the local configuration 001 is at state 1 and that of 100 and 110 is at state 0.

  The transition function of the ECA is a left shift function.

  (a) If the middle cell of the local configuration 011 is at state 0, then the transition function can be represented by Wolfram codes 6, 38, 134 and 166. The ECA will be as shown in Figure 6(a).

  (b) If the middle cell of the local configuration 011 is at state 1, then the transition function can be represented by Wolfram codes 14, 46, 142 and 174. The ECA will be as shown in Figure 6(b).

- Case 2: The middle cell of the local configuration 100 is at state 1 and that of 001 and 011 is at state 0.

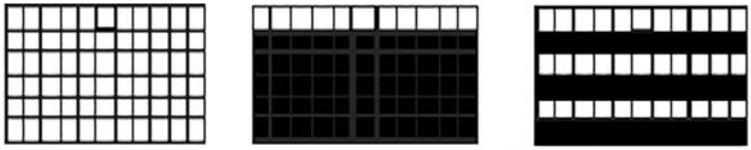
  The transition function of the ECA is a right shift function.

  (a) If the middle cell of the local configuration 110 is at state 0, then the transition function can be represented by Wolfram codes 20, 52, 148 and 180. The ECA will be as shown in Figure 6(c).

  (b) If the middle cell of the local configuration 011 is at state 1, then the transition function can be represented by Wolfram codes 84, 116, 212 and 244. The ECA will be as shown in Figure 6(d).

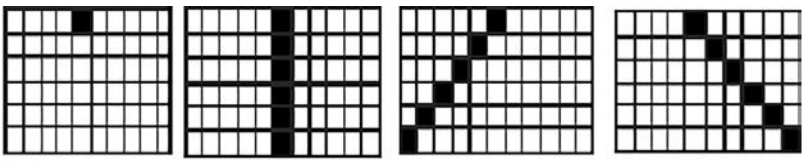
Hence the theorem. □

Any cell of the ECA having state “1” or ON is represented by a black cell and any cell having state “0” or OFF is represented by a white cell in the figures.



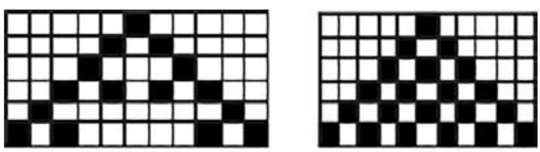
(a) Constant Transition (Rule 0 Equivalent) (b) Constant Transition (Rule 255 Equivalent) (c) Complimentary Transition (Rule 51 Equivalent)

Figure 3. ECA having all cells in the OFF state initially.



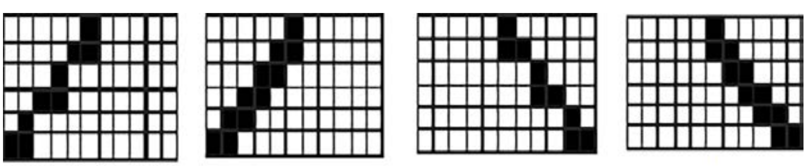
(a) Constant Transition (Rule 0 Equivalent) (b) Identity Transition (Rule 204 Equivalent) (c) 1-place Left Shift Transition (Rule 170 Equivalent) (d) 1-place Right Shift Transition (Rule 240 Equivalent)

Figure 4. ECA starting with exactly one cell in the ON state.



(a) Rule 90 Equivalent (b) Rule 50 Equivalent

Figure 5. ECA never having adjacent ON state cells.



(a) (b) Left Shift Transitions (c) (d) Right Shift Transitions

Figure 6. ECA having shift transitions.

5. Conclusion

The results obtained in this paper are for homogeneous and synchronous cellular automata (CAs). Evolution patterns for elementary CAs having at most one active cell initially have been discussed here. The evolution-wise equivalent Wolfram codes (which induce automorphism) for such elementary CAs having constant, complimentary, identity and shift transitions have been reported. Also evolution-wise equivalent Wolfram codes for rules 90 and 50 have been formulated. An investigation/extension of results for other types of CAs may be worth attempting.

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