Besicovitch Pseudodistances with Respect to Non-Følner Sequences

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The Besicovitch pseudodistance defined in [1] for biinfinite sequences is invariant by translations. We generalize the definition to arbitrary locally compact second-countable groups and study how properties of the pseudodistance, including invariance by translations, are determined by those of the sequence of sets of finite positive measure used to define it. In particular, we restate from [2] that if the Besicovitch pseudodistance comes from an exhaustive Følner sequence, then every shift is an isometry. For non-Følner sequences, it is proved that some shifts are not isometries, and the Besicovitch pseudodistance with respect to some subsequences even makes them discontinuous.

Keywords: Besicovitch distance; Følner sequences; submeasures; amenability; non-compact space; symbolic dynamics

1. Introduction

The Besicovitch pseudodistance was proposed by Blanchard, Formenti and Kůrka in [1] as an "antidote" to sensitivity of the shift map in the prodiscrete (Cantor) topology of the space of one-dimensional configurations over a finite alphabet. The idea is to take a window on the integer line, which gets larger and larger, and compute the probability that in a point under the window, chosen uniformly at random, two configurations will take different values. The upper limit of this sequence of probabilities behaves like a distance, except for taking a value of zero only on pairs of equal configurations: this defines an equivalence relation, and the resulting quotient space is a metric space on which the shift is an isometry, or equivalently, the pseudodistance is shift invariant.

The original choice of windows is $X_n = [\![-n, n]\!]$, the set of integers from $-n$ to n included. This notion can be easily extended to

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arbitrary dimension $d \geq 1$, taking a sequence of hypercubic windows. If we allow arbitrary shapes, the notion of Besicovitch space can be extended to configurations over arbitrary groups; in this case, however, the properties of the group and the choice of the windows can affect the pseudodistance being or not being shift invariant. An example of a Besicovitch pseudodistance that is not shift invariant is given in [2], where it is also proved that, if a countable group is *amenable* (cf. [3] and [4, Chapter 4]), then the Besicovitch pseudodistance with respect to any exhaustive *Følner sequence* is shift invariant. The class of amenable groups is of great interest and importance in group theory, symbolic dynamics and cellular automata theory.

In the last two disciplines we mentioned, the groups are considered by default as discrete spaces, where every subset is open (and closed). A discrete space is always locally compact, and is second countable if and only if it is countable. In addition, the *counting measure* ($\lambda(U) = n$ if U is finite with n elements, $\lambda(U) = \infty$ if U is infinite) has the role of a *Haar measure* such that $\lambda(aU) = \lambda(U)$ for every element a and subset U. But every locally compact group has a Haar measure, unique up to a multiplicative constant, so we may think of applying our discussion to this broader class: the role of the finite nonempty windows from the previous paragraph will be taken by Borel measurable subsets of finite, but positive, measure. The definition of Besicovitch pseudodistance will be similar, provided that we restrict our attention to the class of Borel measurable configurations.

In this paper, which expands and extends our submission [5] to AUTOMATA 2020, we explore the relations between the properties of Besicovitch pseudodistances over configuration spaces with locally compact groups (which we will sometimes suppose to be second countable) and those of the sequence of subsets of finite positive measure used to define it. In Section 3, we give the main definition and prove that if a sequence of subsets of finite positive measure is nondecreasing and has unbounded measure, then the corresponding Besicovitch space is pathwise connected: this generalizes [1, Proposition 1]. In Section 4, we introduce a notion of *synchronous Følner equivalence* between sequences, and a related order relation where one sequence comes before another sequence if it is synchronously Følner equivalent to a subsequence of the latter. This, on the one hand, generalizes Følner sequences, and on the other hand, allows us to compare the Besicovitch pseudodistances and submeasures associated with different sequences. In particular, we prove that an increasing sequence of subsets of finite positive measure is Følner if and only if every shift is an isometry for the corresponding Besicovitch pseudodistance: this provides the converse of [2, Theorem 3.5]. Finally, we give conditions for absolute continuity and Lipschitz continuity of Besicovitch submeasures with respect to each other.

2. Background

Throughout the paper, we will suppose that all the topological spaces discussed are Hausdorff. We call *alphabet* a discrete nonempty finite set.

2.1 Haar Measure

Let G be a locally compact topological group and let Σ be its Borel σ-algebra. It is well known that G admits a Haar measure $\lambda : \Sigma \to [0, \infty]$ satisfying the following properties:

1. $\lambda(G) > 0$.
2. $\lambda(gS) = \lambda(S)$ for every $g \in G$ and $S \in \Sigma$.
3. $\lambda(K) < \infty$ for every compact $K \subseteq G$.
4. For every $S \in \Sigma$, $\lambda(S) = \inf\{\lambda(U) \mid S \subseteq U, \ U \text{ open}\}$.
5. For every open $U \subseteq G$, $\lambda(U) = \sup\{\lambda(K) \mid K \subseteq S, \ K \text{ compact}\}$.

In addition, the Haar measure is unique up to a positive multiplicative constant. From properties 1–5, it follows that $\lambda(U) > 0$ for every nonempty open set U.

Example 1. If G is discrete, the counting measure defined by $\lambda(U) = |U|$ if U is finite and $\lambda(U) = \infty$ if U is infinite is the unique Haar measure on G such that $\lambda(\{g\}) = 1$ for every $g \in G$.

Example 2. If $G = (\mathbb{R}, +)$, the *Lebesgue measure* (more precisely, its restriction to the Borel σ-algebra) is the unique Haar measure on $\mathbb{R}$ such that $\lambda([0, 1]) = 1$.

2.2 Pseudodistances Induced by Submeasures

Definition 1. Let Σ be a σ-algebra of subsets of G. A submeasure over Σ is a map $\mu : \Sigma \to \mathbb{R} \bigsqcup \{+\infty\}$ such that for every $V, W \in \Sigma$:

1. $\mu(\emptyset) = 0$.
2. $\mu(V \cup W) \leq \mu(V) + \mu(W)$.
3. $\mu(V) \leq \mu(W)$ if $V \subseteq W$.

If G and A are two sets, the difference set of two configurations $x, y \in A^G$ is the set $\Delta(x, y) = \{i \in G \mid x(i) \neq y(i)\}$.

Any submeasure over G gives rise to an associated pseudodistance over the set C of measurable configurations $x \in A^G$ (i.e., for every $a \in A$, $\{i \in G \mid x_i = a\} \in \Sigma$): since the difference set is then also measurable, we define:

$$d_\mu(x, y) = \mu(\Delta(x, y)) \forall \ x, y \in C.$$

When G is discrete, all of its subsets are measurable (and even open), so that the reader can forget about the measurability conditions.

Remark 1. The topological space corresponding to such a pseudodistance is homogeneous in the following sense: the balls around every two points y and z are isometric. Indeed, identify A with the additive group $\mathbb{Z} / |A|\mathbb{Z}$. Then for every $y, z \in C$, the map $\psi_{y,z} : C \to C$ defined by $\psi_{y,z}(x)(i) = x(i) - y(i) + z(i)$ for every $x \in C$ and $i \in G$ is an isometry between any ball around y and the corresponding one around z.

We say that submeasure μ is *absolutely continuous* with respect to submeasure ν if $\nu(W) = 0 \Longrightarrow \mu(W) = 0$ for any measurable subset $W \subset G$.

Remark 2. Let $\epsilon, \delta > 0$, μ, ν two submeasures on G, and $z \in C$. The following are equivalent:

1. For every measurable subset $W \subset G$, $\mu(W) \geq \epsilon \Longrightarrow \nu(W) \geq \delta$.
2. For every $x, y \in C$, $d_\mu(x, y) \geq \epsilon \Longrightarrow d_\nu(x, y) \geq \delta$.
3. For every $x \in C$, $d_\mu(x, z) \geq \epsilon \Longrightarrow d_\nu(x, z) \geq \delta$.

Consequently, the identity map, from space C endowed with d_ν onto space C endowed with d_μ, is continuous (resp. α-Lipschitz) if and only if μ is absolutely continuous with respect to ν (resp. $\mu \leq \alpha\nu$). In that case, the identity is even absolutely continuous.

2.3 Shifts and Translations

If A is an alphabet, G is a group and $g \in G$, the *shift* by g is the function $\sigma^g : C \to C$ defined by $\sigma^g(x)(i) = x(g^{-1}i)$ for every $x \in A^G$ and $i \in G$. A map ψ from C to itself is shift invariant if $\psi\sigma^g = \sigma^g\psi$ for every $g \in G$. Note that $\Delta(\sigma^g(x), \sigma^g(y)) = g\Delta(x, y)$ for every $x, y \in C$ and $g \in G$.

Since the maps $\psi_{y,z}$ from Remark 1 are shift invariant, it can be seen that the shift is continuous, Lipschitz, and other properties in every x if and only if it is in one x.

Given a submeasure μ on G and an element g of G, define $g\mu(W) = \mu(g^{-1}W)$ for every measurable subset $W \subset G$. Then the shift by g, within space C endowed with d_μ, is topologically the same as the identity map, from C endowed with d_μ onto space C endowed with $d_{g^{-1}\mu}$, because:

$$\mu(\Delta(\sigma^g(x), \sigma^g(y)) = \mu(g\Delta(x, y)) = g^{-1}\mu(\Delta(x, y)).$$

Remark 1 can then be rephrased into the following.

Remark 3. If G is a group, $g \in G$, and C is endowed with d_μ, then the shift map by g is continuous (resp. α-Lipschitz) if and only if $g^{-1}\mu$ is absolutely continuous (resp. α-Lipschitz) with respect to μ. In that case, the shift by g is even absolutely continuous.

3. Besicovitch Submeasure and Pseudodistance

Among classical examples of submeasures are the ones that induce the Cantor topology, or shift-invariant Besicovitch, or Weyl pseudodistances (see [6, Def 4.1.1]. We will focus on the Besicovitch topology.

3.1 Definition

Measure space. Let G be a nonempty topological space endowed with a Borel measure λ with the following regularity and homogeneity properties:

1. For every set of finite measure W and every $\epsilon > 0$, there is a compact set K such that $\lambda(W \Delta K) < \epsilon$.
2. There exists $\beta > 0$ such that every point has a neighborhood of measure at most β.

These properties are satisfied by several measures, such as the Haar measure if G is a locally compact group: there exists a compact subset of positive measure, which we can call β, and any given element of G belongs to some translate of that subset.

Example 3. Let G be discrete and let λ be the counting measure. Then Point 1 is trivially satisfied as the compacts are precisely the finite subsets, and Point 2 is satisfied with $\beta = 1$.

Our hypotheses allow the following, rather natural, construction.

Lemma 1. For every subset of finite measure $W \subset G$ and every real number $\gamma \in]-\delta, \lambda(W)]$, there exists a Borel set $U \subset W$ such that $\lambda(U) \in \gamma + [0, \beta[$.

Proof. If $\gamma = \lambda(W)$, then simply take $U = W$. Otherwise, the two properties of the measure give a compact subset K such that $\lambda(W \Delta K) < \lambda(W) - \gamma$, and open neighborhoods of measure at most β for each point of K, which cover the whole K. By compactness, K is covered by finitely many of them, say $\tilde{B}_0, \tilde{B}_1, \ldots, \tilde{B}_{k-1}$. Let us consider $B_i = \tilde{B}_i \cap W$, so that $W \cap K \subset \bigcup_{i<k} B_i$. Note that:

- $\lambda\left(\bigcup_{i<0} B_i\right) = \lambda(\emptyset) = 0$.
- $\lambda\left(\bigcup_{i<k} B_i\right) \geq \lambda(W \cap K) \geq \lambda(W) - \lambda(W \Delta K) \geq \gamma$.
- $\lambda\left(\bigcup_{i\leq j} B_i\right) - \lambda\left(\bigcup_{i\leq j} B_i\right) = \lambda(B_j) \in [0, \beta]$ for every $j \in [\![0, k[\![$.

Hence, there exists a minimal $j \in [\![0, k[\![$ such that $\lambda\left(\bigcup_{i\leq j} B_i\right) \geq \gamma$, and we must have $\lambda\left(\bigcup_{i\leq j} B_i\right) < \gamma + \beta$ (note that this is satisfied by $j = 0$ if $\gamma \in]-\beta, 0]$). It is enough to define $U = \bigcup_{i\leq j} B_i$. □

Conditional probability. For all Borel subsets $W, V \subset G$ such that $0 < \lambda(V) < \infty$, let us denote $\lambda(W \mid V) = \frac{\lambda(W \cap V)}{\lambda(V)}$.

Remark 4. Let U, V, W be Borel subsets such that $0 < \lambda(V) < \infty$.

1. $\lambda(W \cup U \mid V) \leq \lambda(W \mid V) + \lambda(U \mid V)$, and the equality holds if the union is disjoint.
2. If $V \subset U$ and $\lambda(U) < \infty$, then
   $\lambda(V \mid U)\lambda(W \mid V) = \lambda(V \cap W \mid U) \leq \lambda(W \mid U)$.

Exhaustive sequences. Let (X_n) be a sequence of subsets of G such that $0 < \lambda(X_n) < \infty$ for every $n \in \mathbb{N}$. (The positive-measure assumption is here for convenience, but is not relevant for the core of the statements.) (X_n) is *nondecreasing* if for every $n \in \mathbb{N}$, $X_n \subseteq X_{n+1}$. It is *exhaustive* if it is nondecreasing and $\bigcup_{n\in\mathbb{N}} X_n = G$.

Remark 5. Consider an exhaustive sequence (X_n), and $W \subset G$ such that $\lambda(W) < \infty$. Then:

1. If $\epsilon > 0$, then there exists $m \in \mathbb{N}$ such that $\lambda(W \setminus X_m) < \epsilon$.
2. If $\lambda(W) > 0$, then there exists $m \in \mathbb{N}$ such that $\lambda(W \mid X_m) > 0$.
3. If $\lambda(G) = \infty$, then for every $\epsilon > 0$, there exists $q_{(X_n)}(W, \epsilon)$ such that for every $n \geq q_{(X_n)}(W, \epsilon)$, $\lambda(W \mid X_n) < \epsilon$ and $\lambda(W \setminus X_n) < \epsilon$.

By nondecreasingness of (X_n), statements of the form "there exists m" can be replaced into the form "there exists n such that for all $m \geq n$."

Proof.

1. $W = \bigsqcup_{n\in\mathbb{N}} W \cap X_{n+1} \setminus X_n$, so that $\lambda(W) = \sum_{n\in\mathbb{N}} \lambda(W \cap X_{n+1} \setminus X_n)$. The convergence of this sum implies that there exists $m \in \mathbb{N}$ such that
   $\lambda(W \setminus X_m) = \sum_{n\geq m} \lambda(W \cap X_{n+1} \setminus X_n) < \epsilon$.
2. Simply pick the m from Point 1 with any $\epsilon < \lambda(W)$.
3. If $\lambda(W) = 0$, there is nothing to prove. Otherwise, since G has infinite measure, it must admit a subset $V \subset G$ of measure at least $\frac{\lambda(W)}{\epsilon} + \epsilon$. By Point 1, there exists $\tilde{q}_{(X_n)}(W, \epsilon)$ such that for all $n \geq \tilde{q}_{(X_n)}(W, \epsilon)$, $\lambda(V \cup W \setminus X_n) < \epsilon$. In particular, $\lambda(X_n) \geq \lambda(V \cup W) - \epsilon \geq \lambda(V) - \epsilon \geq \frac{\lambda(W)}{\epsilon}$.

Hence $\lambda(W \mid X_n) \leq \frac{\lambda(W)}{\frac{\lambda(W)}{\epsilon}} \leq \epsilon$. By taking a maximum with the m obtained in Point 1, we get the desired two inequalities. □

Besicovitch submeasure. The *Besicovitch submeasure* $\mu_{(X_n)}$ is defined over the Borel σ-algebra into [0, 1] by:

$$\mu_{(X_n)}(W) = \limsup_n \lambda(W \mid X_n).$$

The *Besicovitch pseudodistance* is $d_{(X_n)} = d_{\mu_{(X_n)}}$.

Example 4. Let $G = \mathbb{Z}$, $A = \{0, 1\}$ and $X_n = [\![-n, n]\!]$ for every $n \in \mathbb{N}$.

If $x(i) = 0$ for every $i \in \mathbb{Z}$ and $y \in \{0, 1\}^{\mathbb{Z}}$ is the characteristic function of the prime numbers, then $d_{(X_n)}(x, y) = 0$.

Remark 6. In general, we will assume that $\lim_{n\to\infty} \lambda(X_n) = \infty$. In that case, $\mu_{(X_n)}(W) = 0$ for every finite-measure set W.

3.2 Connectedness

Lemma 2. Let (X_n) be nondecreasing such that $\lim_{n\to\infty} \lambda(X_n) = \infty$, $W \subset G$ a Borel set, $\gamma \in [0, 1]$. Then there exists a Borel set $V \subset W$ such that $\mu(V) = \gamma\mu(W)$.

Proof. Let us define, by induction on $n \in \mathbb{N}$, $U_n \subset W \cap X_n$ such that $\lambda(U_n) \in \gamma\lambda(W \cap X_n) + [0, \beta[$ and, if $n > 1$, $U_n \cap X_{n-1} = U_{n-1}$.

First, U_0 can easily be defined as $V \subset W \cap X_0$ by Lemma 1.

Assume that U_n has been so defined, and let us define U_{n+1}. Let $\gamma' = \gamma\lambda(W \cap X_{n+1}) - \lambda(U_n)$. By induction hypothesis, $\gamma' \in \gamma\lambda(W \cap X_{n+1}) - \gamma\lambda(W \cap X_n) - [0, \beta[$, which is equal to $\gamma\lambda(W \cap X_{n+1} \backslash X_n) - [0, \beta[$, since (X_n) is nondecreasing. Since $\gamma \in [0, 1]$, we deduce that $-\beta < \gamma' \leq \lambda(W \cap X_{n+1} \backslash X_n)$. We can apply Lemma 1 to get a subset $U \subset W \cap X_{n+1} \backslash X_n$ such that $\lambda(U) \in \gamma' + [0, \beta[$. Defining $U_{n+1} = U_n \sqcup U$, we get $\lambda(U_{n+1}) = \lambda(U_n) + \lambda(U) \in \lambda(U_n) + \gamma + [0, \beta[= \gamma\lambda(W \cap X_{n+1}) + [0, \beta[$.

We now define $V = \bigcup_{n\in\mathbb{N}} U_n$. By construction, we immediately get that $V \subset W$, and $V \cap X_n = U_n$ for every $n \in \mathbb{N}$. Besides,

$$\lambda(U_n \mid X_n) \in \gamma\lambda(W \cap X_n \mid X_n) + \left[\!\left[0, \frac{\beta}{\lambda(X_n)}\right]\!\right],$$

so that $\lambda(U_n \mid X_n) \sim_{n\to\infty} \gamma\lambda(W \mid X_n)$, since $\lambda(X_n)$ goes to infinity. We get:

$$\begin{aligned}\mu(V) &= \limsup_{n\to\infty} \lambda(V \mid X_n)\\ &= \limsup_{n\to\infty} \lambda(U_n \mid X_n)\\ &= \limsup_{n\to\infty} \gamma\lambda(W \mid X_n)\\ &= \gamma\mu(W). \ \square\end{aligned}$$

Theorem 1. If (X_n) is a nondecreasing sequence such that $\lim_{n\to\infty} \lambda(X_n) = \infty$, then the Besicovitch space is pathwise connected.

Proof. Let $x, y \in C$. Let us define $\phi(0) = x$ and $\phi(1) = y$. Now, for $k \in \mathbb{N}$, suppose that we have defined, for every $(\gamma_i)_{1\le i\le k} \in \{0, 1\}^k$, a Borel set $\phi\left(\sum_{1\le i\le k} \gamma_i 2^{-i}\right)$ of W such that:

$$\mu\left(\Delta\left(\phi\left(\sum_{1\le i\le k} \gamma_i 2^{-i}\right), \phi\left(\sum_{1\le i\le k} \gamma_i 2^{-i} + 2^{-k}\right)\right)\right) = \frac{\mu(\Delta(x, y))}{2^k}.$$

Let

$$(\gamma_i)_{1\le i\le k+1} \in \{0, 1\}^{k+1},$$

$$x' = \phi\left(\sum_{1\le i\le k} \gamma_i 2^{-i}\right), \text{ and}$$

$$y' = \phi\left(\sum_{1\le i\le k} \gamma_i 2^{-i} + 2^{-k}\right).$$

From Lemma 2, there exists a Borel set $V \subset \Delta(x', y')$ such that

$$\mu(V) = \frac{1}{2}\mu(\Delta(x', y')).$$

Define

$$z = \phi\left(\sum_{1\le i\le k+1} \gamma_i 2^{-i}\right)$$

by $z_j = y'_j$ if $j \in V$, and $z_j = x'_j$ if $j \in G\setminus V$. Then

$$\mu(\Delta(x', z)) = \mu(V) = \frac{1}{2}\mu(\Delta(x', y')) = \frac{\mu(\Delta(x, y))}{2^{k+1}}$$

and

$$\mu(\Delta(z, y')) = \mu(\Delta(x', y')) - \mu(V)) = \frac{\mu(\Delta(x, y))}{2^{k+1}}.$$

Since ϕ is a Lipschitz function from the set of dyadic numbers from $[0, 1]$ (endowed with the Euclidean metric) into C, we can extend it into a Lipschitz function from $[0, 1]$ into C. □

4. Følner Equivalence and Besicovitch Submeasures

4.1 Følner Equivalence

The definitions in this subsection are inspired by the notion of Følner sequence [7].

Synchronous Følner equivalence. Let (X_n) and (Y_n) be sequences of subsets of finite positive measure of G. We say that they are *synchronously Følner equivalent* if

$$\lim_{n\to\infty} \frac{\lambda(X_n \Delta Y_n)}{\lambda(X_n)} = 0.$$

Proposition 1. Consider sequences (X_n) and (Y_n) of subsets of finite positive measure. The following are equivalent.

1. (X_n) and (Y_n) are synchronously Følner equivalent.
2. $\lambda(X_n \cap Y_n) \sim_{n\to\infty} \lambda(X_n) \sim_{n\to\infty} \lambda(Y)_n$.
3. $\lambda(X_n) \sim_{n\to\infty} \lambda(Y_n)$ and $\lambda(X_n \backslash Y_n) = o_{n\to\infty}(\lambda(X_n))$.

Proof.

$1 \Longrightarrow 2$ This follows from the obvious inequalities, since X_n and Y_n have finite measure: $\lambda(X) \geq \lambda(X \cap Y) \geq \lambda(X) - \lambda(X \Delta Y)$ and $|\lambda(X) - \lambda(Y)| \leq \lambda(X \Delta Y)$.

$2 \Longrightarrow 3$ Just note that $\lambda(X_n \backslash Y_n) = \lambda(X_n) - \lambda(X_n \cap Y_n)$.

$3 \Longrightarrow 1$ Note that:

$$\begin{aligned} \lambda(X_n \Delta Y_n) &= \lambda(X_n \backslash Y_n) + \lambda(Y_n \backslash X_n) \\ &= \lambda(X_n \backslash Y_n) + \lambda(Y_n) - \lambda(X_n \cap Y_n) \\ &= 2\lambda(X_n \backslash Y_n) + \lambda(Y_n) - \lambda(X_n) \\ &= o_{n\to\infty}(\lambda(X_n)). \end{aligned}$$ □

Corollary 1. Synchronous Følner equivalence is an equivalence relation.

Proof.

- From Proposition 1, note that if (X_n) and (Y_n) are synchronously Følner equivalent, then
$$\frac{\lambda(X_n \Delta Y_n)}{\lambda(Y_n)} \sim_{n\to\infty} \frac{\lambda(X_n \Delta Y_n)}{\lambda(X_n)} \to_{n\to\infty} 0.$$
So the relation is symmetric.

- Transitivity follows from Proposition 1 and from the inclusion $X\Delta Z \subseteq (X\Delta Y) \cup (Y\Delta Z)$, which holds for every finite-measure set X, Y and Z.
- Reflexivity is trivial. □

Since the definition involves a lim (and not a lim inf), we immediately note the following.

Remark 7. (X_n) and (Y_n) are synchronously Følner equivalent if and only if (X_{k_n}) and (Y_{k_n}) are synchronously Følner equivalent, for every increasing sequence (k_n).

Følner dominance. We denote $(X_n) \preceq (Y_n)$, and say that (X_n) is *Følner dominated* by (Y_n), if (X_n) is synchronously Følner equivalent to a subsequence (Y_{k_n}).

Remark 8. $\preceq$ is a preorder relation.

Proposition 2. Let (X_n) be an exhaustive sequence, and (Y_n) a sequence of subsets of finite positive measure such that $\lim_{n\to\infty} \lambda(Y_n) = \infty$.

Then for every $n \in \mathbb{N}$, $m_n = \min_{m\in\mathbb{N}} \lambda(X_n \Delta Y_m)$ is reached, and $\sup_{n\in\mathbb{N}} m_n = +\infty$.

Moreover, $(X_n) \preceq (Y_n)$ if and only if (X_n) is synchronously Følner equivalent to (Y_{m_n}).

By symmetry of synchronous equivalence, this means that $(X_n) \preceq (Y_n)$ if and only if

$$\lim_{n\to\infty} \min_{m\in\mathbb{N}} \frac{\lambda(X_n \Delta Y_m)}{\lambda(X_n)} = \lim_{n\to\infty} \min_{m\in\mathbb{N}} \frac{\lambda(X_n \Delta Y_m)}{\lambda(Y_m)} = 0.$$

Proof.

- For each $n \in \mathbb{N}$, $\lambda(X_n \Delta Y_m) \geq \lambda(Y_m \backslash X_n) \geq \lambda(Y_m) - \lambda(X_n)$ goes to infinity when m goes to infinity. In particular, there is $k \in \mathbb{N}$ such that for every $m \geq k$, $\lambda(X_n \Delta Y_m) \geq \lambda(X_n \Delta Y_0)$. So the minimum of $\lambda(X_n \Delta Y_m)$ (and of $\frac{\lambda(X_n)\Delta Y_m}{\lambda(X_n)}$) is reached between $[\![0, k[\![$.
- Let us prove that $\lim_{n\to\infty} m_n = \infty$. Let $k \in \mathbb{N}$. Since $\sup_{n\in\mathbb{N}} \lambda(Y_n) = \infty$, there exists $l \geq k$ such that for every $m < l$, $\lambda(Y_m) < \lambda(Y_l)$. Since (X_n) is exhaustive, Point 1 of Remark 5 gives some $n \in \mathbb{N}$ such that $\lambda(Y_l \backslash X_n) < \frac{1}{2}(\lambda(Y_l) - \max_{m<l} \lambda(Y_m))$. For every $m < l$,

$$\begin{aligned}\lambda(X_n\Delta Y_m) &\geq \lambda(X_n\backslash Y_m) \text{ because } X_n\Delta Y_m \supseteq X_n\backslash Y_m\\ &\geq \lambda(X_n)-\lambda(Y_m)\\ &\geq \lambda(X_n)-\max_{m<l}\lambda(Y_m)\\ &> \lambda(X_n)+2\lambda(Y_l\backslash X_n)-\lambda(Y_l) \text{ by definition of } n\\ &> \lambda(X_n\Delta Y_l).\end{aligned}$$

We get that $m \neq m_n$, and overall, $m_n \geq l \geq k$ is arbitrarily large.

- By the previous point, (m_n) admits a strictly increasing subsequence. Hence, if (X_n) is synchronously Følner equivalent to (Y_{m_n}), then by definition, $(X_n) \preceq (Y_n)$.
- Conversely, assume that (X_n) is strongly Følner equivalent to some subsequence (Y_{k_n}). By definition of m_n, $\lambda(X_n\Delta Y_{m_n}) \leq \lambda(X_n\Delta Y_{k_n})$. Hence, $\lim_{n\to\infty} \frac{\lambda(X_n\Delta Y_{m_n})}{\lambda(X_n)} \leq \lim_{n\to\infty} \frac{\lambda(X_n\Delta Y_{k_n})}{\lambda(X_n)} = 0.$ □

Følner equivalence. We say that two sequences (X_n) and (Y_n) of subsets of finite positive measure are Følner equivalent, and write $(X_n) \sim (Y_n)$, if both $(X_n) \preceq (Y_n)$ and $(Y_n) \preceq (X_n)$.

This is the case if they are synchronously Følner equivalent, but the converse is false.

Example 5. Let $G = \mathbb{Z}$ and for every $n \geq 1$ let $X_n = [\![-n, -n]\!]$ and $Y_n = X_{1+\lfloor\log n\rfloor}$, where log is the base-2 logarithm. Then

$$\min_{m\geq 1} \frac{\lambda(X_n\Delta Y_m)}{\lambda(X_n)} = 0$$

is reached for $m = 2^n - 1$, and

$$\min_{m\geq 1} \frac{\lambda(Y_n\Delta X_m)}{\lambda(Y_n)} = 0$$

is reached for $m = 1 + \lfloor\log n\rfloor$; by Proposition 2, $(X_n) \sim (Y_n)$. However, $\lambda(Y_n) = o_{n\to\infty}(\lambda(X_n))$, so by Proposition 4, (X_n) and (Y_n) are not synchronously Følner equivalent.

Remark 9. Følner equivalence is an equivalence relation.

Proposition 3. Consider two sequences (X_n) and (Y_n) of subsets of finite positive measure such that (Y_n) is nondecreasing and $\lambda(X_n) \sim_{n\to\infty} \lambda(Y_n)$.

Then (X_n) and (Y_n) are synchronously Følner equivalent if and only if $(X_n) \preceq (Y_n)$.

Proof. Assume that there is a sequence (k_n) such that (X_n) is synchronously Følner equivalent to (Y_{k_n}) (the converse implication is trivial). Let $n \in \mathbb{N}$. If $k_n \leq n$, then $\lambda(X_n \backslash Y_n) \leq \lambda(X_n \backslash Y_{k_n})$ and $\lambda(Y_n \backslash X_n) \leq \lambda(Y_n \backslash Y_{k_n}) + \lambda(Y_{k_n} \backslash X_n)$ since (Y_n) is nondecreasing. Summing up, $\lambda(X_n \Delta Y_n) \leq \lambda(X_n \Delta Y_{k_n}) + \lambda(Y_n \backslash Y_{k_n})$. Symmetrically, if $n \leq k_n$, $\lambda(X_n \Delta Y_n) \leq \lambda(X_n \Delta Y_{k_n}) + \lambda(Y_{k_n} \backslash Y_n)$. Overall, we get:

$$\lambda(X_n \Delta Y_n) \leq \lambda(X_n \Delta Y_{k_n}) + \left|\lambda(Y_{k_n}) - \lambda(Y_n)\right|.$$

By synchronous Følner equivalence, this gives:

$$\lambda(X_n \Delta Y_n) = \left|\lambda(Y_{k_n}) - \lambda(Y_n)\right| + o_{n\to\infty}\lambda(Y_{k_n}).$$

Besides, Proposition 1 (applied to (X_n) and (Y_{k_n})) gives $\lambda(Y_{k_n}) \sim_{n\to\infty} \lambda(X_n) \sim_{n\to\infty} \lambda(Y_n)$. Summing up, we deduce that:

$$\lambda(X_n \Delta Y_n) = o_{n\to\infty}\lambda(X_n). \square$$

4.2 Comparing Besicovitch Submeasures

Point 3 of Remark 5 allows the following construction, which will be useful in the following proofs.

Lemma 3. Let (X_n) be an exhaustive sequence of an infinite-measure G. Let $W = \bigcup_{i\in\mathbb{N}} W_i$ where $W_i \subset G$ has finite positive measure for each $i \in \mathbb{N}$, such that, for every $n \in \mathbb{N}$, $j_n = \max_{W_j \cap X_n \neq \emptyset} j$ is well defined (i.e., X_n intersects finitely many W_i). Then:

1. $\mu_{(X_n)}(W) \geq \limsup_{i\to\infty} \max_{m\in\mathbb{N}} \lambda(W_i \mid X_m)$.
2. If $\lim_{n\to\infty} \epsilon_n = 0$ and $q_{(X_n)}(\bigcup_{i<j_n} W_i, \epsilon_n) \leq n$ for every $n \in \mathbb{N}$, then:
   $\mu_{(X_n)}(W) = \limsup_{i\to\infty} \max_{m\in\mathbb{N}} \lambda(W_i \mid X_m)$.
3. In general, there exists an increasing integer sequence $\mathbb{l}$ such that, denoting $W_{\mathbb{l}} = \bigcup_{i\in\mathbb{N}} W_{l_i}$:
   $\mu_{(X_n)}(W_{\mathbb{l}}) = \lim_{i\to\infty} \max_{m\in\mathbb{N}} \lambda(W_{l_i} \mid X_m)$.

Proof.

1. By Point 2 of Remark 5, there exists $m \in \mathbb{N}$ such that $\lambda(W_i \mid X_m) > 0$. For $i \in \mathbb{N}$, define k_i such that $\lambda(W_i \mid X_{k_i}) = \max_{m\in\mathbb{N}} \lambda(W_i \mid X_m)$. Let us prove that for every $k \in \mathbb{N}$, $\exists\, l \in \mathbb{N}$ such that for every $i > l$, $k_i > k$. The assumption gives that $\{i \in \mathbb{N} \mid W_i \cap X_k \neq \emptyset\}$ is finite, so that we can call l its maximum. In other words, for every $i > l$ and every $m \leq k$, $W_i \cap X_m = \emptyset$ (because (X_n) is nondecreasing), so that

$\lambda(W_i \mid X_m) = 0 < \max_{m\in\mathbb{N}} \lambda(W_i \mid X_m)$, so $k_i > k$. We have proven that $\lim_{n\to\infty} k_i = \infty$. Hence, $\mu_{(X_n)}(W) = \limsup_{n\to\infty} \lambda(W \mid X_n) \geq \limsup_{i\to\infty} \lambda(W \mid X_{k_i})$. We get the desired inequality by noting that $W_i \subset W$.

2. Point 1 already gives one inequality. For the converse:

$$\begin{aligned}\mu_{(X_n)}(W) &= \limsup_{n\to\infty} \lambda\left(\bigcup_{i<j_n} W_i \cup W_{j_n} \cup \bigcup_{i>j_n} W_i \mid X_n\right)\\ &\leq \limsup_{n\to\infty} \lambda\left(\bigcup_{i<j_n} W_i \mid X_n\right)\\ &\quad + \limsup_{n\to\infty} \lambda(W_{j_n} \mid X_n)\\ &\quad + \limsup_{n\to\infty} \lambda\left(\bigcup_{i>j_n} W_i \mid X_n\right)\\ &\leq \limsup_{n\to\infty} \epsilon_n + \limsup_{n\to\infty} \max_{m\in\mathbb{N}} \lambda(W_{j_n} \mid X_m) + 0\\ &\leq 0 + \limsup_{i\to\infty} \max_{m\in\mathbb{N}} \lambda(W_i \mid X_m).\end{aligned}$$

The last inequality comes from the fact that the sequence (j_n) is nondecreasing (because (X_n) is nondecreasing), and not upper bounded (because the W_i are nonempty), so it goes to infinity.

3. Pick any sequence (ϵ_n) converging to 0. Let us define some sequence $\mathbf{l}$ by recurrence, from any seed $l_0 \in \mathbb{N}$. Assume that l_n is defined, and write $k_n = q_{(X_n)}\left(\bigcup_{j\leq n} W_{l_j}, \epsilon_n\right)$. Let $l_{n+1} = j_{k_n-1}$, so that for every $m \geq l_{n+1}$, W_m does not intersect X_{k_n-1}.

The sequence being defined, consider $\tilde{j}_n = \max_{W_{l_j}\cap X_n\neq\emptyset} j$. By definition of the sequence, $k_{\tilde{j}_n-1} = q_{(X_n)}\left(\bigcup_{j<\tilde{j}_n} W_{l_j}, \epsilon_n\right)$, and $W_{l_{\tilde{j}_n}}$ does not intersect $X_{k_{\tilde{j}_n-1}-1}$. Since (X_n) is nondecreasing, it does not intersect any X_m, with $m \leq k_{\tilde{j}_n-1} - 1$. On the other hand, the definition of $\tilde{j}_n$ gives that $W_{l_{\tilde{j}_n}}$ intersects X_n. We can deduce that $n > k_{\tilde{j}_n-1} - 1$. This means that $W_{\mathbf{l}}$ satisfies the hypothesis of Point 2.

Replacing the lim sup by a lim can be achieved by taking, again, a subsequence. □

Now we are able to prove the main equivalence of the paper.

Lemma 4. Let $\epsilon, \delta > 0$, and (X_n), (Y_n) be exhaustive. The following are equivalent.

1. For every $W \subset G$, if $\mu_{(Y_n)}(W) \geq \epsilon$, then $\mu_{(X_n)}(W) \geq \delta$.
2. $\liminf_{n\in\mathbb{N}} \max_{m\in\mathbb{N}} \frac{\epsilon\lambda(Y_n) - \lambda(Y_n \setminus X_m)}{\lambda(X_m)} \geq \delta$.

If k_n realizes the maximum for each $n \in \mathbb{N}$, and if $\epsilon < 1$, then these properties imply that

$$\frac{\delta}{\epsilon} \leq \liminf_{n \in \mathbb{N}} \frac{\lambda(Y_n)}{\lambda(X_{k_n})} \leq \limsup_{n \in \mathbb{N}} \frac{\lambda(Y_n)}{\lambda(X_{k_n})} \leq \frac{1-\delta}{1-\epsilon}.$$

In particular, the properties imply that $\delta \leq \epsilon$.

Proof. Let us start by proving the final inequalities. Assume that

$$\liminf_{n \in \mathbb{N}} \frac{\epsilon\lambda(Y_n) - \lambda(Y_n \backslash X_{k_n})}{\lambda(X_{k_n})} \geq \delta.$$

Then on the one hand, it is clear that

$$\liminf_{n \in \mathbb{N}} \frac{\epsilon\lambda(Y_n)}{\lambda(X_{k_n})}$$

is even bigger, which gives the first inequality. On the other hand, since $\lambda(Y_n \backslash X_{k_n}) \geq \lambda(Y_n) - \lambda(X_{k_n})$, we can see that:

$$\liminf_{n \in \mathbb{N}} (\epsilon - 1) \frac{\lambda(Y_n)}{\lambda(X_{k_n})} + 1 \geq \liminf_{n \in \mathbb{N}} \frac{\epsilon\lambda(Y_n) - \lambda(Y_n \backslash X_{k_n})}{\lambda(X_{k_n})} \geq \delta,$$

which gives

$$\limsup_{n \in \mathbb{N}} \frac{\lambda(Y_n)}{\lambda(X_{k_n})} \leq \frac{1-\delta}{1-\epsilon},$$

provided that $\epsilon < 1$. □

$2 \Rightarrow 1$

$$\begin{aligned}
\mu_{(X_n)}(W) &= \limsup_{m \to \infty} \lambda(W \mid X_m) \\
&\geq \limsup_{n \to \infty} \lambda(W \mid X_{k_n}) \\
&\geq \limsup_{n \to \infty} \frac{\lambda(W \cap Y_n \cap X_{k_n})}{\lambda(X_{k_n})} \\
&= \limsup_{n \to \infty} \frac{\lambda(W \cap Y_n) - \lambda(W \cap Y_n \backslash X_{k_n})}{\lambda(X_{k_n})} \\
&\geq \limsup_{n \to \infty} \frac{\lambda(W \cap Y_n) - \lambda(Y_n \backslash X_{k_n})}{\lambda(X_{k_n})}
\end{aligned}$$

$$
\begin{aligned}
&= \limsup_{n\to\infty}\left(\frac{\epsilon\lambda(Y_n) - \lambda\left(Y_n \backslash X_{k_n}\right)}{\lambda\left(X_{k_n}\right)} + \frac{\lambda\left(W \cap Y_n\right) - \epsilon\lambda(Y_n)}{\lambda(Y_n)}\frac{\lambda(Y_n)}{\lambda\left(X_{k_n}\right)}\right) \\
&\geq \liminf_{n\to\infty}\frac{\epsilon\lambda(Y_n) - \lambda\left(Y_n \backslash X_{k_n}\right)}{\lambda\left(X_{k_n}\right)} \\
&\quad + \left(\limsup_{n\to\infty}\frac{\lambda\left(W \cap Y_n\right)}{\lambda(Y_n)} - \epsilon\right)\liminf_{n\in\mathbb{N}}\frac{\lambda(Y_n)}{\lambda\left(X_{k_n}\right)} \\
&\geq \delta + 0\frac{\delta}{\epsilon}
\end{aligned}
$$

by the two premises and the first inequalities.

$1 \Rightarrow 2$ Let us build a set W that contradicts Point 1, assuming that:

$$\liminf_{n\in\mathbb{N}}\max_{m\in\mathbb{N}}\frac{\epsilon\lambda(Y_n) - \lambda\left(Y_n \backslash X_m\right)}{\lambda(X_m)} \geq \delta.$$

For each $n \in \mathbb{N}$, by Point 1 of Remark 5, there exists $k_n = \min\left\{k \mid \lambda\left(Y_n \backslash X_k\right) \leq \epsilon\lambda(Y_n)\right\}$. By noting that

$$Y_n \cap X_{k_n} \backslash X_{k_n-1} = \left(Y_n \backslash X_{k_n-1}\right) \backslash \left(Y_n \backslash X_{k_n}\right)$$

(by convention X_{-1} is empty), we can write that

$$\lambda\left(Y_n \cap X_{k_n} \backslash X_{k_n-1}\right) = \lambda\left(Y_n \backslash X_{k_n-1}\right) - \lambda\left(Y_n \backslash X_{k_n}\right),$$

which is bigger than $\epsilon\lambda(Y_n) - \lambda\left(Y_n \backslash X_{k_n}\right)$, by minimality of k_n. By Lemma 1, $Y_n \cap X_{k_n} \backslash X_{k_n-1}$ includes a measurable set $Z_n \subset W$ such that

$$\lambda(Z_n) \in \epsilon\lambda(Y_n) - \lambda\left(Y_n \backslash X_{k_n}\right) + [0, \beta[.$$

Define

$$W_n = \left(Y_n \backslash X_{k_n}\right) \bigsqcup Z_n.$$

Note that $W_n \subset Y_n$, and that

$$\epsilon < \lambda(W_n \mid Y_n) \leq \epsilon + \frac{\beta}{\lambda(Y_n)},$$

so that

$$\lim_{n\to\infty}\lambda(W_n \mid Y_n) = \epsilon.$$

The W_i satisfy the hypotheses of Lemma 3 with $j_{k_n} < n$ for every $n \in \mathbb{N}$ (since k_n is unbounded and (X_n) is nondecreasing, this proves that j_m is finite for every m), so that Point 3 gives an integer sequence $\mathbf{l}$, with

$$\mu_{(X_n)}(W_{\mathbf{l}}) = \lim_{i\to\infty} \max_{m\in\mathbb{N}} \lambda\left(W_{l_i} \,\middle|\, X_m\right).$$

By construction, we have:

$$\lambda(W_i \mid X_m) = \frac{\lambda\left(Y_i \cap X_m \backslash X_{k_i}\right) + \lambda\left(Z_i \cap X_m\right)}{\lambda(X_m)}.$$

If $m \le k_i$, then $X_m \subseteq X_{k_i}$, and $Z_i \cap X_m \subseteq Z_i \cap X_{k_i - 1} = \emptyset$, so that $\lambda(W_i \mid X_m) = 0$. On the contrary, if $m > k_i$, then $Z_i \subseteq X_{k_i} \subseteq X_m$, and $Y_i \cap X_m \backslash X_{k_i} = \left(Y_i \backslash X_{k_i}\right) \backslash \left(Y_i \backslash X_m\right)$, so that:

$$\begin{aligned}\lambda(W_i \mid X_m) &= \frac{\lambda\left(Y_i \cap X_m \backslash X_{k_i}\right) + \lambda(Z_i)}{\lambda(X_m)} \\ &\le \frac{\lambda\left(Y_i \backslash X_{k_i}\right) - \lambda\left(Y_i \backslash X_m\right) + \epsilon\lambda(Y_i) - \lambda\left(Y_i \backslash X_{k_i} + \beta\right)}{\lambda(X_m)} \\ &\le \max_{m\in\mathbb{N}} \frac{\epsilon\lambda(Y_i) - \lambda\left(Y_i \backslash X_m\right) + \beta}{\lambda(X_m)}.\end{aligned}$$

The lim inf in both sides of this inequality together with our hypothesis gives

$$\mu_{(X_n)}(W_{\mathbf{l}}) < \delta.$$

On the other hand, applying now Point 1 of Lemma 3 to sequence (Y_n):

$$\begin{aligned}\mu_{(Y_n)}(W_{\mathbf{l}}) &\ge \lim_{i\to\infty} \max_{m\in\mathbb{N}} \lambda\left(W_{l_i} \,\middle|\, Y_m\right) \\ &\ge \lim_{i\to\infty} \lambda\left(W_{l_i} \,\middle|\, Y_{l_i}\right) = \lim_{n\to\infty} \lambda(W_n \mid Y_n) = \epsilon. \;\square\end{aligned}$$

The previous lemma allows us to now characterize the main properties of interest for comparing two Besicovitch submeasures.

Proposition 4. Let (X_n) and (Y_n) be exhaustive sequences.

1. For each $\alpha > 0$, $\mu_{(Y_n)} \le \alpha\mu_{(X_n)}$ if and only if

$$\forall\, \epsilon > 0,\ \liminf_{n\to\infty} \max_{m\in\mathbb{N}} \frac{\lambda(Y_n) - \frac{1}{\epsilon}\lambda(Y_n \backslash X_m)}{\lambda(X_m)} \ge \frac{1}{\alpha}.$$

2. $\mu_{(Y_n)}$ is absolutely continuous with respect to $\mu_{(X_n)}$ if and only if $\exists\, \alpha > 0,\ \mu_{(Y_n)} \leq \alpha\mu_{(X_n)}$.
3. $\mu_{(Y_n)} \leq \mu_{(X_n)}$ if and only if $(Y_n) \preceq (X_n)$.
4. $\mu_{(Y_n)} = \mu_{(X_n)}$ if and only if $(Y_n) \sim (X_n)$.

We can even see from the proof that $(Y_n) \preceq (X_n)$ if and only if there exists $\epsilon \in]0, 1[$ such that for every Borel measurable subset $W \subset G$, $\mu_{(X_n)}(W) < \epsilon \Longrightarrow \mu_{(Y_n)}(W) < \epsilon$.

Proof.

1. Just note that $\mu_{(Y_n)} \leq \alpha\mu_{(X_n)}$ is equivalent to the properties in Lemma 4, for every δ and $\epsilon = \alpha\delta$, and hence to:
$$\liminf_{n\in\mathbb{N}} \max_{m\in\mathbb{N}} \frac{\lambda(Y_n) - \frac{1}{\epsilon}\lambda(Y_n \backslash X_m)}{\lambda(X_m)} \geq \frac{1}{\alpha}.$$
2. From Lemma 4, $\mu_{(Y_n)}$ is absolutely continuous with respect to $\mu_{(X_n)}$ if and only if
$$\forall\, \epsilon > 0,\ \liminf_{n\to\infty} \max_{m\in\mathbb{N}} \frac{\lambda(Y_n) - \frac{1}{\epsilon}\lambda(Y_n \backslash X_m)}{\lambda(X_m)} > 0.$$
From Point 1, this is equivalent to the existence of some α such that $\mu_{(Y_n)} \leq \alpha\mu_{(X_n)}$.
3. Consider a sequence (k_n) that witnesses that $(Y_n) \preceq (X_n)$:
$$\lim_{n\to\infty} \frac{\lambda(Y_n \Delta X_{k_n})}{\lambda(Y_n)} = 0.$$
By Point 3 of Proposition 1,
$$\lim_{n\in\mathbb{N}} \frac{\lambda(Y_n) - \frac{1}{\epsilon}\lambda(Y_n \backslash X_{k_n})}{\lambda(X_{k_n})} = \lim_{n\in\mathbb{N}} \frac{\lambda(Y_n)}{\lambda(X_{k_n})}\left(1 - \frac{1}{\epsilon}\lim_{n\in\mathbb{N}} \frac{\lambda(Y_n \backslash X_{k_n})}{\lambda(Y_n)}\right)$$
$$= 1\left(1 - \frac{1}{\epsilon}0\right).$$
We can conclude with Point 1. Conversely, suppose that
$$\liminf_{n\in\mathbb{N}} \frac{\lambda(Y_n) - \frac{1}{\epsilon}\lambda(Y_n \backslash X_{k_n})}{\lambda(X_{k_n})} \geq 1.$$
By the last inequalities in Lemma 4, we know that
$$\lim_{n\in\mathbb{N}} \frac{\lambda(Y_n)}{\lambda(X_{k_n})} = 1.$$

Moreover,

$$\lim_{n\to\infty} \frac{\lambda(Y_n \backslash X_{k_n})}{\lambda(X_{k_n})} \leq \lim_{n\to\infty} \frac{\epsilon\lambda(Y_n)}{\lambda(X_{k_n})}$$
$$-\epsilon \liminf_{n\in\mathbb{N}} \frac{\lambda(Y_n) - \frac{1}{\epsilon}\lambda(Y_n \backslash X_{k_n})}{\lambda(X_{k_n})}$$
$$= \epsilon - \epsilon = 0.$$

By Point 3 of Proposition 1, we obtain that $(Y_n) \preceq (X_n)$.

4. This statement follows directly from the definitions and Point 3. □

The following is direct from Proposition 4 and Remark 2.

Corollary 2. Let (X_n) and (Y_n) be exhaustive. Then $(Y_n) \preceq (X_n)$ (resp. $(Y_n) \sim (X_n)$) if and only if the identity map from space C endowed with $d_{(X_n)}$ onto space C endowed with $d_{(Y_n)}$ is 1-Lipschitz (resp. an isometry).

Here are particular classes of sequences, where the proposition can be applied.

Corollary 3. Let (X_n) and (Y_n) be exhaustive.

1. If there exist a real number $\alpha > 0$ and a sequence (k_n) such that

   $$\liminf_{n\to\infty} \lambda(Y_n \mid X_{k_n}) \geq \frac{1}{\alpha}$$

   and $Y_n \subset X_{k_n}$, then $\mu_{(Y_n)} \leq \alpha\mu_{(X_n)}$.

2. On the other hand, if $\lambda(X_n) \sim_{n\to\infty} \lambda(Y_n)$ but (X_n) and (Y_n) are not (synchronously) Følner equivalent, and $q_{(Y_m)}(X_n, \epsilon_n) = n+1$ for some real sequence (ϵ_n) converging to 0, then $\mu_{(X_n)}$ is not absolutely continuous with respect to $\mu_{(Y_n)}$.

Proof.

1. We can use Point 1 of Proposition 4, because for every $\epsilon > 0$,

$$\liminf_{n\to\infty} \max_{m\in\mathbb{N}} \frac{\lambda(X_n) - \frac{1}{\epsilon}\lambda(X_n \backslash Y_m)}{\lambda(Y_m)} \geq \liminf_{n\to\infty} \frac{\lambda(X_n) - \frac{1}{\epsilon}\lambda(X_n \backslash Y_{k_n})}{\lambda(Y_{k_n})}$$
$$= \liminf_{n\to\infty} \frac{\lambda(X_n)}{\lambda(Y_{k_n})}$$
$$\geq \frac{1}{\alpha}.$$

2. Suppose $\lambda(X_n) \sim_{n\to\infty} \lambda(Y_n)$ and (X_n) and (Y_n) are not synchronously Følner equivalent. By Proposition 3, $(X_n) \not\lesssim (Y_n)$, that is,

$$\epsilon = \limsup_{n\to\infty} \frac{\lambda(X_n \backslash Y_n)}{\lambda(Y_n)} > 0.$$

We can write:

$$\liminf_{n\to\infty} \frac{\lambda(X_n) - \frac{1}{\epsilon}\lambda(X_n \backslash Y_n)}{\lambda(Y_n)} = 0.$$

By the second assumption, for every $m > n$, $X_n \backslash Y_m = \emptyset$ and $\frac{\lambda(X_n)}{\lambda(Y_m)} \leq \epsilon_n$. We get:

$$\max_{m\in\mathbb{N}} \frac{\lambda(X_n) - \frac{1}{\epsilon}\lambda(X_n \backslash Y_m)}{\lambda(Y_m)} \leq \max\left(\frac{\lambda(X_n) - \frac{1}{\epsilon}\lambda(X_n \backslash Y_n)}{\lambda(Y_n)}, \epsilon_n\right).$$

Putting things together, $\liminf_{n\to\infty} \max_{m\in\mathbb{N}} \frac{\lambda(X_n) - \frac{1}{\epsilon}\lambda(X_n \backslash Y_m)}{\lambda(Y_m)}$ is 0. We conclude with Point 2 of Proposition 4. □

Corollary 4. Let (X_n) and (Y_n) be exhaustive. Assume that $\lambda(X_n) \sim_{n\to\infty} \lambda(Y_n)$. Then the following are equivalent.

1. (X_n) and (Y_n) are (synchronously) Følner equivalent.
2. $\mu_{(Y_{l_n})} = \mu_{(X_{l_n})}$, for every increasing sequence (l_n).
3. $\mu_{(Y_{l_n})}$ is absolutely continuous with respect to $\mu_{(X_{l_n})}$, for every increasing sequence (l_n).

Proof.

$1 \Longrightarrow 2$ By Remark 7, synchronous Følner equivalence is transmitted to all subsequences (provided that one takes the same subsequence for (X_n) and for (Y_n)). We conclude using Proposition 4.

$2 \Longrightarrow 3$ This is obvious.

$\not{1} \Longrightarrow \not{3}$ If (X_n) and (Y_n) are not synchronously Følner equivalent, then there exists an infinite set $I \subset \mathbb{N}$ and a real number $\alpha > 0$ such that $\forall\, n \in I$,

$$\frac{\lambda(X_n \Delta Y_n)}{\lambda(X_n)} \geq \alpha.$$

This implies that for every increasing sequence $(l_n) \in I^{\mathbb{N}}$, (X_{l_n}) and (Y_{l_n}) are not synchronously Følner equivalent. We can take an increasing sequence $(l_n) \in I^{\mathbb{N}}$ such that $q_{(Y_m)}(X_{l_n}, \epsilon_{l_n}) = l_{n+1}$, for some real sequence (ϵ_n) converging to 0. Then (X_{l_n}) and (Y_{l_n}) satisfy the assumptions for Point 2 of Corollary 3. □

4.3 Shift

Throughout this subsection and the next two, G will be a locally compact group with Haar measure λ, and (X_n) will be a sequence of subsets of G such that $0 < \lambda(X_n) < \infty$ for every $n \in \mathbb{N}$. If G is discrete, the Haar measure of choice is the counting measure. Countable groups are presumed discrete.

Følner sequences. (X_n) is *(left) g-Følner* if $(X_n) \sim (gX_n)$. Since $\lambda(X_n) = \lambda(gX_n)$, Proposition 3 says that it is enough to require $(X_n) \preceq (gX_n)$, and that in this case, (X_n) and (gX_n) are synchronously Følner equivalent. (X_n) is a *(left) Følner sequence* if it is g-Følner for every $g \in G$.

Finitely generated groups. A countable group G is *finitely generated* (briefly, f.g.) if a finite subset $E \subset G$ exists such that for every $g \in G$ there exists $e_1, \ldots, e_n \in E \bigcup E^{-1}$ such that $e_1 \cdots e_n = g$. The minimum such n is called the *norm* of g (w.r.t. E) and denoted by $\|g\|$; the set $\{g \in G \mid \|g\| \leq n\}$ is called the *ball of radius n* (w.r.t. E).

If the size of the balls grows polynomially with the radius, then they form a Følner sequence, so Point 3 of Corollary 5 generalizes [6, Cor 4.1.4]. Vice versa, if the size of the balls grows exponentially with the radius, then no subsequence of the sequence of balls is Følner; whether the converse implication holds is still an open problem.

Amenable groups. Let $L^\infty(G)$ be the space of the Borel measurable functions $u : G \to \mathbb{R}$ for which $M \geq 0$ exists such that $\lambda(\{x \in G \mid |u(x)| > M\}) = 0$. G is *amenable* if it has a *left-invariant mean*, that is, a linear functional $m : L^\infty(G) \to \mathbb{R}$ such that:

1. If $f(x) \geq 0$ for every $x \in G$, then $m(f) \geq 0$.
2. If $f(x) = 1$ for every $x \in G$, then $m(f) = 1$.
3. For every $g \in G$ and $f \in L^\infty(G)$, $m(gf) = m(f)$, where gf is defined by $gf(x) = f(g^{-1}x)$ for every $x \in G$.

If G is *σ-countable*, that is, a countable union of compact subsets (in particular, if it is second countable) then (cf. [8, Section 16], in particular Propositions 16.10 and 16.16) G is amenable if and only if it has a Følner sequence, and the sequence can be taken to be exhaustive. For the case when G is discrete, the reader can compare [4, Chapter 4] and [9, Chapter 5].

The following comes directly from Corollary 2.

Corollary 5. Let G be a locally compact group admitting an exhaustive sequence (X_n).

1. The following are equivalent:
   (a) (X_n) is g-Følner.
   (b) $\mu_{(X_n)} = \mu_{(g^{-1}X_n)}$.
   (c) The shift by g is an isometry for $d_{(X_n)}$.
2. (X_n) is Følner if and only if every shift is an isometry for $d_{(X_n)}$.
3. If G is σ-countable (in particular, if it is second countable) then it is amenable if and only if there exists an exhaustive sequence (X_n) such that every shift is an isometry for $d_{(X_n)}$.

Note that one implication of Point 3 was already stated in [2, Theorem 3.5] in the case of countable groups, but the proof contains a confusion between left and right Følner. The full equivalence generalizes [6, Cor 4.1.4].

Corollary 6. Let G be a locally compact group.

1. If G is finitely generated and (X_n) is the sequence of balls with respect to some finite generating set, then the shift by any element $g \in X_{\|g\|}$ is $|X_{\|g\|}|$-Lipschitz.
2. Let $g \in G$. An exhaustive sequence is g-Følner if and only if all of its subsequences induce a Besicovitch pseudodistance for which the shift by g is continuous.
3. If G is second countable, then G is amenable if and only if it admits an exhaustive sequence of which all subsequences yield a Besicovitch pseudodistance that makes every shift continuous.

The first point generalizes [6, Prop 4.1.3]. Note that it still applies in nonamenable groups such as the *free group on two generators*, but the shifts are no longer isometries, and there is a subsequence of balls with respect to which the Besicovitch pseudodistance makes them discontinuous.

Proof.

1. We can apply Point 1 of Corollary 3: $Y_n = g \cdot X_n \subset X_{\|g\|} \cdot X_n = X_{n+\|g\|}$.
2. This comes from Corollary 4 and Remark 3.
3. This comes from Point 2 and the characterization of amenability through Følner sequences. □

There are nondecreasing non-Følner sequences for which the shift is Lipschitz (but not an isometry) in $\mathbb{Z}^d$.

Example 6. Let $G = \mathbb{Z}^d$ endowed with the counting measure, $X_n = (\llbracket -n, n\rrbracket \cup 2\llbracket -n, n\rrbracket)^d$, so that $\left|X_n\right| = (2n + 1 + 2\lceil n/2\rceil)^d \sim_{n\to\infty} (3n)^d$. Let $g \in \mathbb{Z}^d$ such that $\|g\|_\infty = 1$. Then on the one hand, for every n, $X_n + g \subset X_{2n}$ and

$$\frac{|X_{2n}|}{|X_n|} \sim_{n\to\infty} \frac{(6n)^d}{(3n)^d} = 2^d.$$

Point 1 of Corollary 3, with $k_n = 2n$, gives that every shift is 2^d-Lipschitz. On the other hand, the sequence is not g-Følner, because

$$\mu\left((2\mathbb{Z})^d\right) = \limsup_{n\to\infty} \frac{(2n+1)^d}{(3n)^d} = \frac{2^d}{3^d}$$

while

$$\mu\left((2\mathbb{Z}+g)^d\right) = \limsup_{n\to\infty} \frac{n^{\|g\|_1}(2n+1)^{d-\|g\|_1}}{(3n)^d} = \frac{2^{d-\|g\|_1}}{3^d}.$$

By Point 1 of Corollary 5, the shift by g is not an isometry.

4.4 Propagations and Right-Følner Sequences

As the Haar measure is left invariant, the "dual" measure $\overline{\lambda}(U) = \lambda(U^{-1})$ for every Borel set U is *right* invariant. The two measures need not be equal: those groups for which they are, are called *unimodular*; the class of unimodular groups includes discrete groups and amenable groups. The definition of synchronous Følner equivalence, Følner dominance and Følner equivalence can be stated again for the right-Haar measure; we can denote the "dualized" notions as $\preceq_R$ and $\sim_R$. All the results proven for λ have a dual version for $\overline{\lambda}$, provided that multiplication on the left is replaced by multiplication on the right, and vice versa.

Let G be a locally compact group and $g \in G$. A sequence (X_n) of subsets of positive finite measure of G is *right g-Følner* if $(X_n) \sim_R (X_n g)$, that is, (X_n^{-1}) is left g^{-1}-Følner. Equivalently (dualizing Proposition 3), $\overline{\lambda}(X_n \Delta X_n g) = o_{n\to\infty}\overline{\lambda}(X_n)$. A *right-Følner sequence* is then a sequence that is right g-Følner for every $g \in G$.

Now let A be an alphabet. The *propagation* in direction $g \in G$ is the function $\pi^g : C \to C$ defined by $\pi^g(x)(i) = x(ig)$ for every $x \in C$ and $i \in G$. With this definition, the value of $\pi^g(x)$ at point ig equals the value of x at point i: that is, the information moves in direction g. Points 1, 2 and 3 of Corollary 5 can then be *dualized* to right-Følner sequences and propagations:

Corollary 7. Let G be a unimodular locally compact group.

1. An exhaustive sequence (X_n) is right g-Følner, if and only if $\mu_{(X_n)} = \mu_{(X_n g)}$ if and only if the propagation in direction g is an isometry.
2. An exhaustive sequence is right Følner if and only if every propagation is an isometry.
3. If G is σ-countable (in particular, if it is second countable) then G is amenable if and only if it admits an exhaustive sequence of which all subsequences yield a Besicovitch pseudodistance that makes every shift continuous.

Proof. Given $x \in C$, let $\overline{x}(i) = x\left(i^{-1}\right)$ for every $x \in C$ and $i \in G$: then for every $x, y \in C$ and $g \in G$, $\overline{\overline{x}} = x$, $\Delta\left(\overline{x}, \overline{y}\right) = (\Delta(x, y))^{-1}$ and $\overline{\pi^g(x)} = \sigma^g(\overline{x})$, thus also $d_{(X_n)}(x, y) = d_{\left(X_n^{-1}\right)}\left(\overline{x}, \overline{y}\right)$ and $d_{(X_n)}(\pi^g(x), \pi^g(y)) = d_{\left(X_n^{-1}\right)}\left(\sigma^g(\overline{x}), \sigma^g(\overline{y})\right)$, so that the propagation in direction g is an isometry for $d_{(X_n)}$ if and only if the shift by g is an isometry for $d_{X_n^{-1}}$. All points then follow easily from Corollary 5. □

4.5 Block Maps

Let G be a discrete group. A *block map* on G with *source alphabet* A, *target alphabet* B, *neighborhood* $N = \{j_1, \ldots, j_k\}$ and *local rule* $\phi : A^k \to B$ is a function $F : A^G \to B^G$ defined as the synchronous application of ϕ at the "N-shaped neighborhood" of each point of the group: that is, for every $x \in A^G$ and $i \in G$, $F(x)(i) = \phi(x(ij_1), \ldots, x(ij_n))$. By the *Curtis–Lyndon–Hedlund theorem* [10] (see also [4, Chapter 1]), block maps are all and only those functions from A^G to B^G that are continuous in the prodiscrete topology and commute with all the shifts. Every propagation is a block map, but the shift by $g \in G$ is a block map if and only if g is *central* in G, that is, $gh = hg$ for every $h \in G$; in this case, $\sigma^g = \pi^{g^{-1}}$. Note that the local rule ϕ can be seen as the local rule of another block map Φ with source alphabet A^k, target alphabet B, and neighborhood $N = \{e\}$, where e is the identity element of G. Since the neighborhood is trivial, we have $\Delta(\Phi(x), \Phi(y)) \subseteq \Delta(x, y)$ for every $x, y \in \left(A^k\right)^G$. Hence Φ is 1-Lipschitz, but not necessarily an isometry: for example, ϕ could be constant.

Block maps can be defined equivalently as follows. For $f_1, \ldots, f_k : A^G \to A^G$ define the *product* $f = f_1 \times \cdots \times f_k : A^G \to \left(A^k\right)^G$

by $f(x)(i) = (f_1(x)(i), \ldots, f_k(x)(i))$ for every $x \in A^G$ and $i \in G$. Then a block map F with source alphabet A, target alphabet B, neighborhood $N = \{j_1, \ldots, j_k\}$ and local rule ϕ has the form $F = \Phi \circ (\pi^{j_1} \times \cdots \times \pi^{j_k})$, where Φ is a block map with source alphabet A^k, target alphabet B, neighborhood $N = \{e\}$, and local rule ϕ, and π_j is the projection to the j^{th} component.

Lemma 5. Suppose $f_1, \ldots, f_k : A^G \to A^G$ are such that f_q is α_q-Lipschitz with respect to $d_{(X_n)}$. Then $f = f_1 \times \cdots \times f_k : A^G \to (A^k)^G$ is $\left(\sum_{q=1}^k \alpha_q\right)$-Lipschitz with respect to $d_{(X_n)}$. In particular, if $\alpha_q = \alpha$ for every $q \in [\![1, k]\!]$, then f is $k\alpha$-Lipschitz, and if each f_q is an isometry, then f is k-Lipschitz.

Proof. For every $x, y \in C$ and $i \in G$, we have $f(x)(i) \neq f(y)(i)$ if and only if $f_q(x)(i) \neq f_q(y)(i)$ for at least one $q \in [\![1, k]\!]$, that is,

$$\Delta(f(x), f(y)) = \bigcup_{q=1}^k \Delta(f_q(x), f_q(y)).$$

Consequently, $\mu_{(X_n)}(\Delta(f(x), f(y)) \leq \sum_{q=1}^k \mu_{(X_n)}(\Delta(f_q(x), f_q(y)))$, and the thesis follows easily. □

The composition of an α-Lipschitz function with a β-Lipschitz function is an $\alpha\beta$-Lipschitz function. As every propagation is a block map, Lemma 5 and Point 2 of Corollary 7 allow us to dualize Points 2 and 3 of Corollary 6 and recover (cf. [2, Theorem 3.7] and [11, Theorem 18]) the following characterization.

Corollary 8. Let G be a discrete group.

1. An exhaustive sequence is right Følner if and only if all of its subsequences yield a Besicovitch pseudodistance that makes every block map continuous.
2. If G is σ-countable (in particular, if it is second countable) then it is amenable if and only if it admits an exhaustive sequence (X_n) such that, for every $k \geq 1$ and every increasing (l_n), every block map with neighborhood size k is k-Lipschitz with respect to $d_{(X_{l_n})}$.

5. Conclusion

We have generalized Besicovitch submeasures from the class of discrete groups to the class of locally compact groups and presented ways to compare them in terms of absolute continuity, Lipschitz continuity and equality, based on the properties of the sequences of subsets of finite positive Haar measure that describe them. Endowing the set of measurable configurations with the Besicovitch topology, we have derived conditions on the defining sequence for the shift maps to be continuous, Lipschitz or isometric. As part of this, we gave another characterization of amenable groups.

Possible future work could involve extension to configuration spaces on possibly non-second-countable groups. This would require the use of the more general notions of *directed set* and of *net*, and although the definition of Besicovitch pseudodistance and submeasure would be immediate to extend, the iterative constructions of Lemmas 2 and 4 could need a major revision.

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