The Buffon Needle Problem Revisited in a Pedagogical Perspective

Ivar G. Johannesen

Imagine a floor marked with many equally spaced parallel lines and a thin stick whose length exactly equals the distance $L = 1$ between the lines. If we throw the stick on the floor, the stick may or may not cross one of the lines. The probability for a hit involves π. This is surprising since there are no circles involved; on the contrary, there are only straight lines. If we repeat the experiment many times and keep track of the hits, we can get an estimate of the irrational number π. (We also consider sticks of length $L > 1$.)

The problem can easily be done as an exercise in a first calculus course, where the students are challenged to consider concepts such as probability, definite integration, symmetry, and inverse trigonometric functions. The solution to this problem therefore gives many applications in a variety of fields in calculus.

We continue by throwing regular polygons of different sizes, increasing the number of edges, and at last reach the ultimate goal of throwing circular objects. This article illustrates the process of throwing sticks, polygons, and circles analytically and graphically, and how to carry out calculations for different n-gons. The result always involves the number π, except when the circle is introduced! We also show the circle result as a limiting value as n increases to infinity.

■ Introduction

The problem of throwing sticks on a set of parallel equidistant lines was first raised by the French naturalist and mathematician Georges Louis Leclerc Comte de Buffon in 1733 and later solved in 1777 by Buffon himself. Despite the apparent linearity of the situation, the result gives us a method for computing the irrational number π. For more than 250 years, scientists have been intrigued by this puzzle, as can be seen by a quick search on the internet. Many authors have extended the exercise to throwing regular polygons. This article considers regular polygons with either even or odd n. When the number of vertices is even, opposite vertices are situated on the diameter of the circumscribed circle. There are no diametrically opposed vertices in odd regular polygons; therefore, these n-gons are more challenging for students to handle. The length L of the stick is replaced by the diameter L of the circumscribed circle when regular polygons are considered.

This article illustrates the process of throwing sticks, polygons, and circles analytically and graphically, and how to carry out calculations for different n-gons. The mathematics necessary are elementary and suitable for students in a first calculus course. The students will solve the necessary integrals and calculate the probabilities by hand before invoking *Mathematica*.

The introductory part of the lab considers sticks of length $L = 1$—the same unit length as the distance between lines. The idea is described in [1, 2], including relevant *Mathematica* code for illustrations. Each throw can be fully described by two parameters: the distance y from the center of the stick to the nearest line, and the acute angle θ that the stick makes with any parallel line (Figure 1).

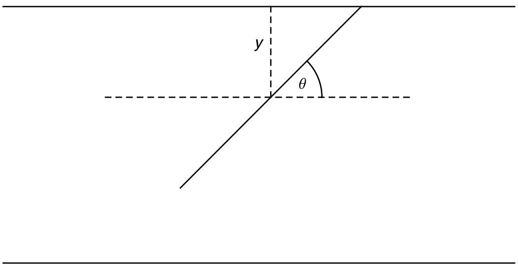


Figure 1. The stick hits the line if $y \leq \frac{1}{2} \sin\theta$.

In the parameter space (θ, y), the graph of the function $y = \frac{1}{2}\sin\theta$ is the border line between the areas representing hits and misses. Due to symmetry, we need only consider $0 \leq y \leq \frac{1}{2}$, $0 \leq \theta \leq \frac{\pi}{2}$ (Figure 2).

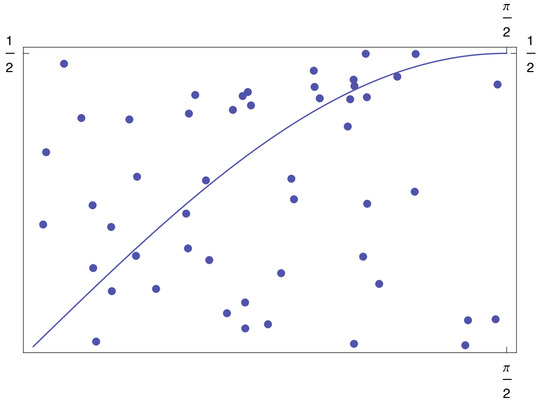


Figure 2. The misses are drawn in gray and the hits in black.

Relating to the main topic in this article, we regard the stick as a degenerate polygon with two vertices and reformulate our function expression accordingly.

The probability of hitting a line is the ratio of the area under the graph to the area of the parameter space.

In[1]:=
```
P[hit] = ∫_0^(π/2) (1/2) Cos[π/2 - θ] dθ / (π/4)
```

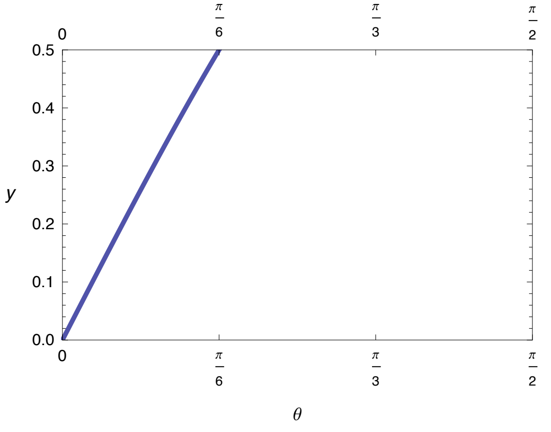
Out[1]= $\frac{2}{\pi}$

This result is interesting because it suggests a way to estimate the number π. Let a group of students draw parallel, equidistant lines on a large piece of paper and throw a substantial number of sticks on it, keeping a record of the hits. If n needles hit a line out of t tries, then the students get an approximation $\frac{2t}{n} \approx \pi$.

■ Long Sticks

Let us look at sticks with arbitrary length L. When $L \leq 1$, the probability of hits is directly proportional to L. When $L > 1$, large values of θ always give hits. Here is the situation for $L = 2$.

In[2]:=
```
Plot[Cos[π/2 - θ], {θ, 0, π/3}, PlotRange → {{0, π/2}, {0, 1/2}},
  Frame → True, FrameLabel → {Style[θ, Larger], Style[y, Larger]},
  RotateLabel → False, PlotStyle → Thickness[0.01],
  FrameTicks → {{0, π/6, π/3, π/2}, Automatic}]
```

Out[2]=


For $\theta \geq \frac{\pi}{6}$, the stick of length 2 always crosses a line because it is inclined so much. On the other hand, arbitrarily long sticks can avoid hitting a line if the inclination is small enough. The probability for a stick of length L is given by the function `probSticks`.

```
In[3]:= probSticks[L_] = 4/π If[L < 1, ∫_0^(π/2) L/2 Sin[θ] dθ,
          (1/2 (π/2 - ArcSin[1/L]) + ∫_0^(ArcSin[1/L]) L/2 Sin[θ] dθ)];
```

The expression is interesting for several reasons. First, we have a "real" situation in which an inverse trigonometric function arises naturally. Second, the definite integral that makes up the last term is noteworthy in that finding an antiderivative is easy, while evaluating it at the integral's endpoints requires a little more work. The students are encouraged to simplify $\cos\left(\arcsin \frac{1}{L}\right)$ and verify the following simpler expression.

```
In[4]:= probSticks[L_] := If[L ≤ 1, 2 L/π, 2/π ((L - Sqrt[L^2 - 1]) + ArcCos[1/L])]
```

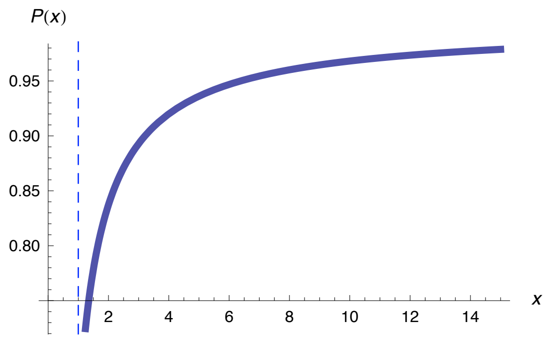
The probabilities always involve the factor $\frac{1}{\pi}$. For $L \leq 1$, the graph is linear.

```
In[5]:= probSticks [1]

Out[5]= 2/π

In[6]:= N[%]

Out[6]= 0.63662
```

The results can be summarized for sticks of any length by plotting the probability of hitting a line as a function of L.

```
In[7]:= Plot[probSticks[L], {L, 0, 15},
          PlotStyle → Thickness[0.015], Epilog →
           {Blue, Dashing[{0.02}], Line[{{1, 0}, {1, 1}}]}, AxesLabel →
           TraditionalForm /@ {Style[x, Larger], Style[P[x], Larger]}]
```

Out[7]=



■ Tossing Squares

We start our investigation of regular polygons by tossing squares on the ruled floor. Let θ be the acute angle between the vertical and a line through the square's center and the midpoint of an edge (Figure 3). Other choices for the angle are also possible.

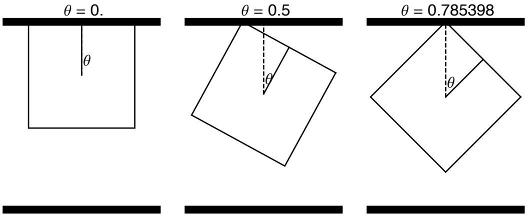


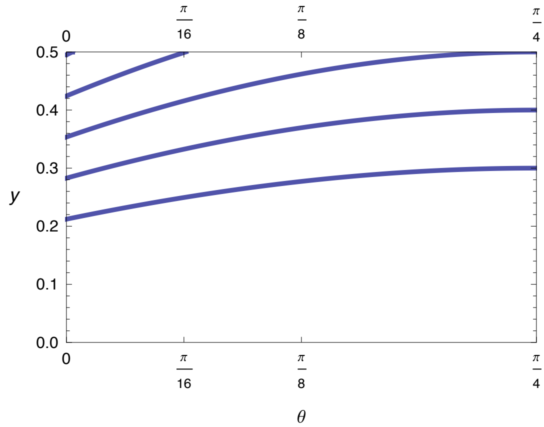
Figure 3. Here are some configurations where the square just touches the line.

The graph in the parameter space dividing hits and misses is given by $y = \frac{L}{2}\cos\left(\frac{\pi}{4} - \theta\right)$, where L is the length of the square's diagonal. This is the same as the diameter of the circumscribed circle. Due to symmetry, it is enough to consider $0 \le \theta \le \frac{\pi}{4}$.

For $L \le 1$, we always have $y \le \frac{1}{2}$. What about squares whose diameter is greater than 1? Since the θ parameter is restricted to $\left[0, \frac{\pi}{4}\right]$, we must consider the limit $L = \sqrt{2}$. If L increases beyond that value, there are always hits with at least one side of the square. Then the curve dividing hits and misses is greater than $1/2$ for all values of θ, and the plot in parameter space is empty. When $L > 1$, the polygon hits a line if $\frac{L}{2}\cos\left(\frac{\pi}{4} - \theta\right) < \frac{1}{2}$ and $\theta < \frac{\pi}{4}$. This means that $\theta > \frac{\pi}{4} - \arccos\left(\frac{1}{L}\right)$.

In[8]:=
```
Show[Table[Plot[1/2 L Cos[π/4 - θ], {θ, 0, If[L ≤ 1, π/4, π/4 - ArcCos[1/L]]},
   PlotRange → {{0, π/4}, {0, 1/2}}, Frame → True,
   FrameLabel → {Style[θ, Larger], Style[y, Larger]},
   RotateLabel → False, PlotStyle → Thickness[0.01],
   FrameTicks → {{0, π/16, π/8, π/4}, Automatic},
   DisplayFunction → Identity], {L, 0.6, 1.4, 0.2}],
 DisplayFunction → $DisplayFunction]
```

Out[8]=



The area in parameter space corresponding to hits is

$$\int_0^{\frac{\pi}{4}} \frac{L}{2} \cos\left(\frac{\pi}{4} - \theta\right) d\theta = \int_0^{\frac{\pi}{4}} \frac{L}{2} \cos\theta \, d\theta$$

for $L \leq 1$, and

$$\frac{1}{2}\left(\frac{\pi}{4} - \left(\frac{\pi}{4} - \arccos\left(\frac{1}{L}\right)\right)\right) + \int_0^{\frac{\pi}{4} - \arccos\left(\frac{1}{L}\right)} \frac{L}{2} \cos\left(\frac{\pi}{4} - \theta\right) \theta =$$

$$\frac{1}{2} \arccos\left(\frac{1}{L}\right) + \int_{\text{arcCos}\left(\frac{1}{L}\right)}^{\frac{\pi}{4}} \frac{L}{2} \cos\theta \, d\theta$$

for $1 < L < \sqrt{2}$. This gives us the probability function

In[9]:=
```
probSquare[L_] :=
 8/π Piecewise[{{L/2 ∫_0^(π/4) Cos[θ] dθ, 0 ≤ L ≤ 1}, {(1/2 ArcCos[1/L] +
      L/2 ∫_(ArcCos[1/L])^(π/4) Cos[θ] dθ), 1 < L ≤ √2}, {π/8, True}}]
```

In[10]:=
```
{probSquare[1], probSquare[Sqrt[2]]}
```

Out[10]= $\left\{\frac{2\sqrt{2}}{\pi}, 1\right\}$

In[11]:=
```
N[%]
```

Out[11]=
```
{0.900316, 1.}
```

Hexagons

Throwing hexagons follows the same outline as squares. Opposite vertices lie on the diameter of the circumscribed circle, and so we have symmetry about $\theta = \frac{\pi}{6}$.

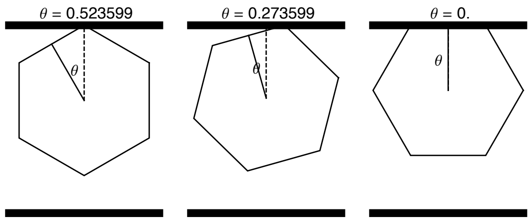


Figure 4. Here are some hexagon configurations.

For $L > 1$, the hexagon hits a line when $\theta \geq \frac{\pi}{6} - \arccos\left(\frac{1}{L}\right)$. For $L \geq \frac{2}{\sqrt{3}}$, at least one line is always hit (Figure 4).

In[12]:=
$$\text{probHexagon[L\_] :=}\ \frac{12}{\pi}\,\text{Piecewise}\left[\left\{\left\{\frac{L}{2}\int_0^{\frac{\pi}{6}}\text{Cos}[\theta]\,d\theta,\ 0 \le L \le 1\right\},\ \left\{\left(\frac{1}{2}\,\text{ArcCos}\left[\frac{1}{L}\right] + \frac{L}{2}\int_{\text{ArcCos}\left[\frac{1}{L}\right]}^{\frac{\pi}{6}}\text{Cos}[\theta]\,d\theta\right),\ 1 < L \le \frac{2}{\sqrt{3}}\right\},\ \left\{\frac{\pi}{12},\ \text{True}\right\}\right\}\right]$$

In[13]:=
```
{probHexagon[1], probHexagon[2/Sqrt[3]]}
```

Out[13]= $\left\{\frac{3}{\pi}, 1\right\}$

Octagons

We continue with octagons (Figure 5).

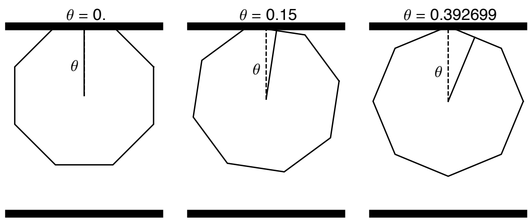


Figure 5. Here are some octogon configurations.

For $L > 1$, the octagon hits a line when $\theta \geq \frac{\pi}{8} - \arccos\left(\frac{1}{L}\right)$. For $L \geq \frac{1}{\cos\left(\frac{\pi}{8}\right)}$, at least one line is always hit.

In[14]:=
```
Off[N::meprec]
```

In[15]:=
```
probOctagon[L_] :=
```
$$\frac{16}{\pi}\,\text{Piecewise}\left[\left\{\left\{\frac{L}{2}\int_0^{\frac{\pi}{8}}\text{Cos}[\theta]\,d\theta,\ 0 \leq L \leq 1\right\},\ \left\{\left(\frac{1}{2}\,\text{ArcCos}\left[\frac{1}{L}\right] + \frac{L}{2}\int_{\text{ArcCos}\left[\frac{1}{L}\right]}^{\frac{\pi}{8}}\text{Cos}[\theta]\,d\theta\right),\ 1 < L \leq \sqrt{4 - 2\sqrt{2}}\right\},\ \left\{\frac{\pi}{16},\ \text{True}\right\}\right\}\right]$$

In[16]:=
$$\left\{\text{probOctagon}[1],\ \text{probOctagon}\left[\frac{1}{\text{Cos}\left[\frac{\pi}{8}\right]}\right]\right\}$$

Out[16]=
$$\left\{\frac{8\,\text{Sin}\left[\frac{\pi}{8}\right]}{\pi},\ 1\right\}$$

In[17]:=
$$\text{probOctagon}[1]\ /.\ \text{Sin}[x\_] \to \sqrt{\frac{1 - \text{Cos}[2\,x]}{2}}\ //\ \text{Simplify}$$

Out[17]=
$$\frac{4\sqrt{2 - \sqrt{2}}}{\pi}$$

In[18]:=
```
N[%]
```

Out[18]=
```
0.974495
```

■ Dodecagons

Before treating the general 2 n-gon, here is the case of the dodecagon (Figure 6).

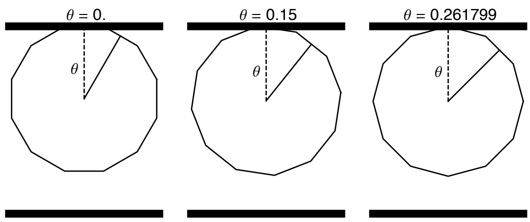


Figure 6. Here are some dodecagon configurations.

In[19]:=
```
probDodecagon[L_] :=
 24/π Piecewise[{{L/2 ∫_0^(π/12) Cos[θ] dθ, 0 ≤ L ≤ 1}, {(1/2 ArcCos[1/L] +
     L/2 ∫_(ArcCos[1/L])^(π/12) Cos[θ] dθ), 1 < L ≤ 2 Sqrt[2 - Sqrt[3]]}, {π/24, True}}]
```

In[20]:=
```
{probDodecagon[1], probDodecagon[1/Cos[π/12]]}
```

Out[20]= $\left\{\frac{3\sqrt{2}\left(-1+\sqrt{3}\right)}{\pi},\ 1\right\}$

In[21]:=
```
N[%]
```

Out[21]= {0.988616, 1.}

■ 2 *n*-gons

For higher-order n-gons where n is even, we encounter the same sort of symmetry about $\theta = \frac{\pi}{n}$ and always get a hit when $L \geq \frac{1}{\cos\left(\frac{\pi}{n}\right)}$.

In[22]:=
```
probNgon[n_?EvenQ, L_] :=
 2 n/π Piecewise[{{L/2 ∫_0^(π/n) Cos[θ] dθ, 0 ≤ L ≤ 1},
   {(1/2 ArcCos[1/L] + L/2 ∫_(ArcCos[1/L])^(π/n) Cos[θ] dθ), 1 < L ≤ 1/Cos[π/n]},
   {π/(2 n), True}}]
```

```
In[23]:= {probNgon[20, 1], probNgon[20, 1/Cos[π/20]]} (* icosagon *)

Out[23]= {20 Sin[π/20]/π, 1}
```

■ Tossing Triangles

Next we look at equilateral triangles. In a regular odd polygon, the adjacent vertices do not lie on the diameter of the circumscribed circle. This means that we must take the full distance between lines into consideration.

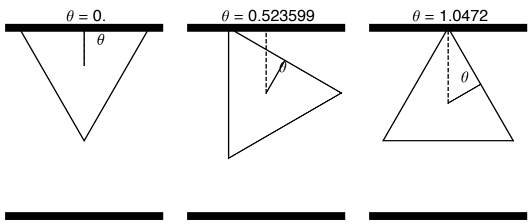


Figure 7. Here are some triangle configurations.

Let y be the vertical distance from the top line to the center of the triangle. This is where the medians cross; the medians are also the altitudes since the triangle is regular. From Figure 7, we see that the border line between the hit and miss areas is $y = \frac{L}{2}\cos\left(\frac{\pi}{3} - \theta\right)$, where L is the diameter of the circumscribed circle. This means that the altitudes (medians) have length $\frac{3L}{4}$. For $L > \frac{4}{3}$, the triangle therefore has to cut one or more lines.

But there is another border line, as Figure 8 shows.

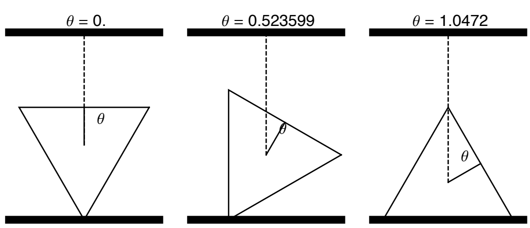


Figure 8. For $y > 1 - \frac{L}{2}\cos(\theta)$, the triangles cut the lower line.

Thus the hit area consists of two distinct parts in parameter space.