# Properties of the Vacuum.
# I. Mechanical and Thermodynamic*

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Casimir energies are calculated for quantized fields in cavities with a variety of forms. Consequences for models of the vacuum state are considered. The possibility of negative mass systems is discussed. Results on energy and entropy of finite quantum systems at non-zero temperature are given.

## 1. Introduction

This is the first in a series of papers on the "bulk" properties of quantized fields. These properties are determined from the "response" of the "vacuum" state to classical external fields or constraints. Such investigations complement perturbative treatments of small disturbances in quantized fields.

This paper discusses some "mechanical" and "thermodynamic" properties of quantized fields. Sections 2 through 6 consider the mechanical forces associated with containment of quantized fields in finite regions. Sections 7 and 8 discuss thermodynamic properties of fields in finite regions at non-zero temperatures. In later papers, we shall consider "electrodynamic" [1] and "gravitational" properties of quantized fields. We shall for the most part use the "free" approximation in which interactions of the fields with themselves are ignored.

The mechanical properties of the vacuum are unlike those of ordinary matter. As mentioned in Section 5, a region of vacuum may under certain circumstances have a negative energy, so that "empty" cavities may apparently attain negative masses.

The vacuum (ground) states of several highly non-linear field theories may have a foam-like structure with "bubbles" of low field strength separated by walls of high field strength. It is plausible that fluctuations of the field within a bubble may exert forces which determine the shape of its walls. We discuss this possibility in section 6 and show that the equilibrium shapes of such bubbles may be isotropic or tubular.

In section 8 we consider the relation between the entropy and energy density of quantized fields at finite temperatures, and show that a recently suggested entropy bound [2] is incorrect.

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We shall usually consider fields contained in regions of space some of whose directions are finite in extent. In many cases, the fields may be viewed as confined by hyper-cuboidal (hyper-rectangular-parallelipedal) boundaries. The necessary constraints on the fields are implemented either by modifying the spacetime on which field exists (so that, for example, certain coordinates have finite ranges, or are periodically identified) or through physical boundary conditions as would result from interaction with extended external sources of infinite strength. Such constraints modify the zero-point modes of the fields, and change the energy of the vacuum (ground state). This change is manifest as an observable Casimir energy [3]. The Casimir energy may be either positive or negative. The corresponding mechanical pressure exerted on the walls of a "container" may be either positive or negative, and is often anisotropic. The result depends on the exact form of the constraints and on the nature of the field.

Section 2 describes in detail the calculation of Casimir energy for the simple case of a scalar (spin 0) field between two plates. It is convenient and illuminating to take the dimensionality of spacetime as an arbitrary continuous parameter. In this way, divergences associated with high frequency fluctuations in the field are regularized.

Sections 3 and 4 generalize the results to confining regions of other shapes, and to vector (spin 1) fields.

Attractive van der Waals "dispersion" forces between electrically neutral macroscopic objects may be attributed to Casimir energies of the electromagnetic field (e.g. [4]). However, as discussed in section 5, not all Casimir energies may be associated with sums of van der Waals forces. For example, the electromagnetic Casimir energy of a three-dimensional cubic cavity with perfectly-conducting walls leads to an outward pressure [5–8], while a sum of van der Waals forces would suggest inward pressure. Sections 3 and 4 consider a variety of systems exhibiting repulsive Casimir forces. The existence and nature of these examples appears to have caused confusion in previous investigations.

## 2. CASIMIR ENERGY IN A SIMPLE SYSTEM

This section describes in some detail the calculation of the Casimir energy for the simple case of a non-interacting scalar field $\varphi$ with mass $m$ in $d$-dimensional space with one direction of finite length $\alpha$. The field satisfies the free Klein–Gordon equation

$$(\partial^2 + m^2)\,\varphi(x) = 0 \tag{2.1}$$

away from any boundaries. One form of constraint on the field is achieved by introduction of explicit boundaries consisting of $d-1$-dimensional hyperplanes located at $x = 0$ and $x = a$. For points $z$ on these boundaries, the field obeys either

$$\varphi(z) = 0 \qquad \text{(Dirichlet)} \tag{2.2a}$$

or

$$\partial_z \varphi(z) = 0 \qquad \text{(Neumann)}. \tag{2.2b}$$

or a linear combination of these (Robin). The Neumann condition (2.2b) is analogous to "bag" boundary conditions, and implies that the momentum flux of the field through the boundaries vanishes. Instead of introducing physical boundaries, one may require the periodicity condition[1] (compactification of $R^1$ to $S^1$)

$$\varphi(0) = \varphi(a). \tag{2.3}$$

We consider first the case of Dirichlet boundary conditions. The modes of the field are then

$$\varphi(x, \mathbf{x}_T, t) = \sin(n\pi x/a)\, e^{i\mathbf{k}_T \cdot \mathbf{x}_T} e^{-i\omega_k t},$$
$$\omega_k = \sqrt{(n\pi/a)^2 + \mathbf{k}_T^2 + m^2}, \tag{2.4}$$

where $n$ is a positive integer. In the ground state (vacuum) each of these modes contributes an energy $\omega_k/2$.[2] For normalization purposes, we take the transverse coordinates $\mathbf{x}_T$ to be restricted by $|\mathbf{x}_T| < L$ where $L \gg a$. The total energy of the field between the planes is thus given by[3]

$$E = (L/2\pi)^{d-1} \int d^{d-1}\mathbf{k}_T \sum_{n=1}^{\infty} \tfrac{1}{2}\omega_k. \tag{2.5}$$

High modes render this sum formally divergent. The contributions of such modes should however be independent of $a$. They should thus cancel in calculations of forces, or in comparisons of the energy density of the field in the presence and the absence of the planes. The sum in Eq. (2.5) may be regularized by a variety of techniques. The simplest is by analytic continuation in $d$. Using the result (e.g. [11])

$$\int f(k)\, d^d k = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int k^{d-1} f(k)\, dk \tag{2.6}$$

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[1] The "antiperiodic" or "twisted" condition $\varphi(0) = -\varphi(a)$ is also consistent.

[2] One of the many formal derivations of this result is given in Sect. 7. So long as non-linear (self) interactions are absent [9], the equal weighting of modes would also follow from a classical field with random amplitude and phases constrained to have a Lorentz invariant spectrum [10]. The normal ordering used to remove disconnected vacuum diagrams in the usual formulation of quantum field theory applies only in an infinite volume: different normal ordering prescriptions must be used in finite volumes, allowing Casimir effects.

[3] If the boundary conditions are charge-conjugation invariant, any density of quantum numbers must cancel between particle and antiparticle modes. Energy is the unique quantity which is positive for both positive and negative frequency modes.

(2.5) becomes

$$E = (L/2\pi)^{d-1} \frac{2\pi^{(d-1)/2}}{\Gamma((d-1)/2)} \sum_{n=1}^{\infty} \int_0^{\infty} \frac{1}{2} (k_T^2)^{(d-3)/2} \, d(k_T^2) \frac{1}{2} \sqrt{(n\pi/a)^2 + k_T^2 + m^2}.$$

(2.7)

The relation

$$\int_0^{\infty} t^r (1+t)^s \, dt = B(1+r, -s-r-1)$$

(2.8)

then yields

$$E = \frac{1}{2} \frac{\Gamma(-d/2)}{\Gamma(-1/2)} \pi^{(d+1)/2} \frac{(L/2)^{d-1}}{a^d} \sum_{n=1}^{\infty} [(am/\pi)^2 + n^2]^{d/2}.$$

(2.9)

We consider first the case $m = 0$. The necessary sum is then formally given by

$$\sum_{n=1}^{\infty} n^d = \zeta(-d),$$

(2.10)

where $\zeta(s)$ is the usual Riemann $\zeta$ function. Using the reflection formula (e.g. [12])

$$\Gamma\left(\frac{s}{2}\right) \pi^{-s/2} \zeta(s) = \Gamma\left(\frac{1-s}{2}\right) \pi^{(s-1)/2} \zeta(1-s)$$

(2.11)

together with the reduplication formula

$$\Gamma(s) \sqrt{\pi} = 2^{s-1} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{1+s}{2}\right)$$

(2.12)

the energy (2.9) may be written

$$E = -\frac{L^{d-1}}{a^d} \Gamma\left(\frac{d+1}{2}\right) (4\pi)^{-(d+1)/2} \zeta(d+1).$$

(2.13)

This result is finite for all positive $d$, and is always negative. The force per unit area on the boundary plates is attractive for any value of $d$ and of magnitude

$$\frac{-\partial(E/L^{d-1})}{\partial a} = -\frac{d}{a^{d+1}} \Gamma\left(\frac{d+1}{2}\right) (4\pi)^{-(d+1)/2} \zeta(d+1).$$

(2.14)

The pressure of the vacuum between the plates is thus negative. In the limit $a \to \infty$, (2.13) gives the energy of the field in the absence of the plates. The regularization by

analytic continuation in $d$ used here sets this energy to zero. Special cases of the result (2.13) are

$$d = 1, \quad E = -\frac{\pi}{48}\frac{1}{a} \simeq -0.065/a,$$

$$d = 2, \quad E = -\frac{\zeta(3)}{16\pi}\frac{L}{a^2} \simeq -0.024L/a^2, \tag{2.15}$$

$$d = 3, \quad E = -\frac{\pi^2}{1440}\frac{L^2}{a^3} \simeq -0.0069L^2/a^3.$$

For small $d$, $E \sim -1/d$; the dimensionless coefficient in $E$ decreases for intermediate $d$, but tends to $-\infty$ as $d \to \infty$, taking on a minimum value of $-2.5 \times 10^{-6}$ at $d \sim 25.2$.

An alternative scheme for regularizing the sum (2.9) is to impose an (exponential) cutoff $\Lambda$ in longitudinal momentum, as for electromagnetic modes in material media (e.g. [3]). Writing $\lambda = \pi/a\Lambda$, the analogue of Eq. (2.10) is

$$\sum_{n=1}^{\infty} n^d e^{-\lambda n} = Li_{-d}(e^{-\lambda}) = e^\lambda \Phi(e^{-\lambda}, -d, 1)$$

$$= \Gamma(1+d)\lambda^{-(d+1)} + e^\lambda \sum_{j=0}^{\infty} \zeta(-d-j)\frac{(-\lambda)^j}{j!}, \tag{2.16}$$

where $\Phi$ is the Lerch transcendent [12]. When $\Lambda \to \infty$ this reduces to (2.10), except for a singular term whose dependence on $a$ is cancelled by the factor $a^d$ in (2.9). As mentioned above, terms independent of $a$ have no physical significance, and may differ between regularization schemes. Identical $a$-dependent terms are always obtained in momentum-cutoff and dimensional continuation schemes. Each term in the sum (2.16) is positive, and the result is correspondingly positive. However, the physically relevant $a$-dependent piece in the energy need not be positive.

The cutoff in longitudinal momentum should be distinguished from a cutoff at a fixed mode quantum number $n$. The latter scheme cannot be implemented through properties of the boundaries. If used, it would yield a divergent $a$-dependent energy.

Another method consists in writing the sum $\sum_{n=1}^{\infty} \omega_k$ as $\lim_{s \to -1} \sum_{n=1}^{\infty} \omega_k^{-s}$. Analytic continuation of the resulting $\zeta$ function then yields a form identical to (2.10) [13, 14].

Equation (2.5) is for fields satisfying Bose–Einstein statistics. The sign is reversed for the case of Fermi–Dirac statistics. When equal numbers of fermion and boson states exist (as in supersymmetric models), the leading divergent contributions to (2.5) cancel. The total vacuum energy of an infinite volume vanishes when corresponding boson and fermion states have equal masses (for the divergent part

alone to cancel, only the sums of the $m^2$ and $m^4$ need be equal [15]). The total Casimir energy in a finite volume will not vanish except with special boundary conditions (such as the periodic ones discussed below).[4] (With bag boundary conditions, or in a spherical Einstein universe [17], the total Casimir energy does not vanish.)

The sum (2.5) directly gives the energy for a field which exists in a cavity, and vanishes outside. The field could instead exist everywhere, but vanish on two thin plates. It is convenient for regularization to enclose such a system in a large cavity. The total energy of the system is then a sum of contributions from the resulting three spaces. With dimensional regularization, the energy of the field in the outer spaces goes to zero as their size goes to infinity. With other regularization schemes, a divergent contribution may remain.

We now treat the case of a massive scalar field. The analytic continuation of the sum (2.9) for non-zero $m$ is derived in the Appendix. The result for the energy is

$$E = -\frac{1}{2} \frac{L^{d-1}}{a^4} (4\pi)^{-(d+1)/2} \left[ -\sqrt{\pi} \Gamma\left( -\frac{d}{2} \right) (am)^d \right.$$
$$\left. + (am)^{(d+1)} \left( \Gamma\left( -\frac{d+1}{2} \right) + 4 \sum_{n=1}^{\infty} \frac{K_{(d+1)/2}(2amn)}{(amn)^{(d+1)/2}} \right) \right], \tag{2.17}$$

where $K$ is a modified Bessel function. The first term in brackets gives a contribution to the total energy independent of $a$ and is therefore dropped. The first term in parentheses corresponds to a constant energy density and would occur even in the absence of the planes. It is cancelled by addition of a constant to the Hamiltonian density. The finite physically relevant energy is thus (cf. [18])

$$E = -\frac{L^{d-1}}{a^d} (4\pi)^{-(d+1)/2} (ma)^{(d+1)} \sum_{n=1}^{\infty} \frac{K_{(d+1)/2}(2amn)}{(amn)^{(d+1)/2}}$$
$$= -\frac{L^{d-1}}{a^d} (4\pi)^{-(d+1)/2} \frac{\pi^{(d+1)/2}}{2} \int_0^{\infty} dt \, t^{(d-1)/2} e^{-(am)^2/nt} (\vartheta(0; t) - 1)$$
$$= -\frac{L^{d-1}}{a^d} (4\pi)^{-(d+1)/2} \sqrt{\pi} 2^{(d+1)/2} \int_0^{\infty} dt \, \frac{J_{(d+1)/2}(t) \, t^{-(d+1)/2}}{e^{\pi t/ma} - 1}$$
$$= -\frac{L^{d-1}}{a^d} \frac{1}{2} (2\pi)^{-d/2} \int_0^{\infty} dt \, t^{d-1} \log(1 - e^{-2\sqrt{t^2 + (ma)^2}}), \tag{2.18}$$

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[4] This is analogous to the fact that supersymmetry is broken in finite temperature systems because boson fields satisfy periodic boundary conditions in imaginary time, while fermion fields satisfy antiperiodic ones [16]. Both boson and fermion fields may, however, satisfy periodic boundary conditions in space.

where $\vartheta$ is a Jacobi theta function (see Appendix), and $J$ is a regular Bessel function.[5] Numerical results for (2.18) (obtained using the second form given) are shown in Fig. 2.1. For small $m$,

$$E \simeq -\frac{L^{d-1}}{a^d} (4\pi)^{-(d+1)/2}$$

$$\times \left[ \Gamma\left(\frac{d+1}{2}\right) \zeta(d+1) - \Gamma\left(\frac{d-1}{2}\right) \zeta(d-1)(am)^2 + \cdots \right] \quad (d \geqslant 3). \tag{2.19}$$

Low modes may be considered responsible for the dependence of the Casimir energy on $a$. When $ma \gg 1$, the energies of these low modes are dominated by $m$ and approximately independent of $a$, so that the physical Casimir energy decreases:

$$E \sim -\frac{L^{d-1}}{a^d} \frac{1}{2} \left(\frac{am}{4\pi}\right)^{d/2} e^{-2ma}. \tag{2.20}$$

As $d$ increases, the contribution of transverse momenta become more important, and $E$ decreases less rapidly with $ma$, as seen in Fig. 2.1.

For the Neumann boundary condition (2.2b), the sin appearing in the modes (2.4) becomes cos, but the final results (2.13) and (2.18) for the physical energy $E$ remain the same.
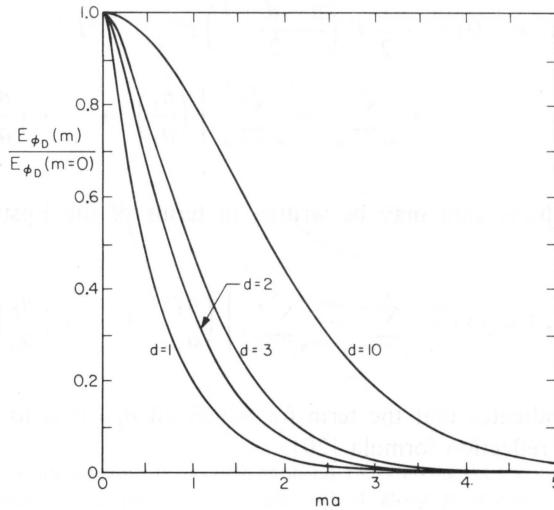


FIG. 2.1.   Casimir energy density for a mass $m$ scalar field satisfying Dirichlet boundary conditions on two planes a distance $a$ apart in $d$ space dimensions.

[5] The third form given is an example of the Walfisz formula discussed in [19].

With the periodicity condition (2.3) the modes of the field become

$$\varphi(x, \mathbf{x}_T, t) = e^{2\pi i n(x/a)} e^{i\mathbf{k}_T \mathbf{x}_T} e^{-i\omega_k t},$$

$$\omega_k = \sqrt{(2n\pi/a)^2 + \mathbf{k}_T^2 + m^2} \tag{2.21}$$

and the energy in this case is $E_{\varphi_P}(a) = 2E_{\varphi_D}(a/2)$, where $E_{\varphi_D}(a)$ denotes the physical energy (2.18) for Dirichlet boundary conditions.

## 3. Casimir Energy in More Complicated Regions

This section derives the Casimir energy for a scalar field in a general hyper-cuboidal region, with $p$ sides of finite length $a_1,..., a_p$ and $d - p$ sides with length $L \gg a_i$.

The simplest results are obtained by taking the field periodic in the finite length directions (case $P$), corresponding to compactification of $p$ space dimensions to a hypertorus $T^p$. The modes of the field then consist of a simple product of modes analogous to (2.21) for each direction. The total energy is obtained by summing separately over the $n_i$ for each set of modes:

$$E_{\varphi_P}(a_1,..., a_p; p; d; m = 0) = -\frac{1}{2} \Gamma\left(\frac{p - d - 1}{2}\right) \pi^{(d-p+1)/2} L^{d-p} \tag{3.1}$$

$$\times \sum_{n_1 = -\infty}^{\infty} \cdots \sum_{n_p = -\infty}^{\infty} \left[\left(\frac{n_1}{a_1}\right)^2 + \cdots + \left(\frac{n_p}{a_p}\right)^2\right]^{(d-p+1)/2}.$$

The necessary multiple sum may be written in terms of the Epstein zeta function [20, 21]

$$Z_p(1/a_1,..., 1/a_p; s) = \sum_{n_1 = -\infty}^{\infty} \cdots \sum_{n_p = -\infty}^{\infty}{}' \left[\left(\frac{n_1}{a_1}\right)^2 + \cdots + \left(\frac{n_p}{a_p}\right)^2\right]^{-s/2}, \tag{3.2}$$

where the prime indicates that the term for which all $n_i = 0$ is to be omitted. This function obeys the reflection formula [20]

$$W_p(a_1,..., a_p; s) = \Gamma\left(\frac{s}{2}\right) \pi^{-s/2} Z_p(a_1,..., a_p; s)$$

$$= a_1^{-1} \cdots a_p^{-1} \Gamma\left(\frac{p - s}{2}\right) \pi^{(s-p)/2} Z_p(1/a_1,..., 1/a_p; p - s) \tag{3.3}$$

analogous to (2.11) and derived in the Appendix. Using (3.3) the Casimir energy (3.1) becomes

$$E_{\varphi_p}(a_1,..., a_p; p; d; m = 0)$$

$$= -\frac{1}{2} L^{d-p} a_1 \cdots a_p \Gamma\left(\frac{d+1}{2}\right) \pi^{-(d+1)/2} Z_p(a_1,..., a_p; d+1). \qquad (3.4)$$

This result is again finite for all $d \geqslant p$ and $p > 0$. It is always negative.

The asymptotic formulae given in the Appendix show that when one of the lengths $a_1,..., a_p$ becomes much larger than the others, the energy tends exponentially to that obtained with only $p - 1$ finite $a_i$.

When all the $a_i = a$ some simplification may occur. Figure 3.1 gives a plot of $W_p(s) = W_p(a_1 = 1,..., a_p = 1; s)$ in this case for several values of $p$.[6] The corresponding physical energies for a few values of $d$ are listed in the first column of Table I. Notice that if all $a_i$ are constrained to be equal, the Casimir energy is negative and leads to a force which tends to contract the system.
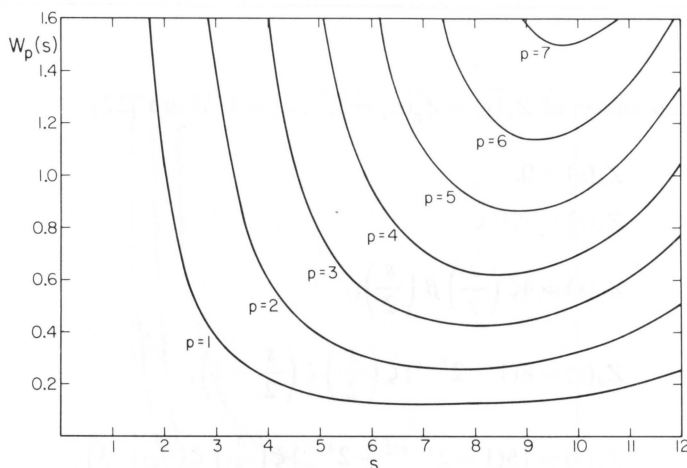


FIG. 3.1. Values of the function $W_p(s)$ defined in Eq. (3.3), giving the Casimir energy of a massless scalar field constrained to be periodic with unit period along $p$ orthogonal directions in $s - 1$-dimensional space.

[6] Accurate numerical evaluation of the necessary sums is easily achieved by a direct summation in which progressively larger sets of terms are averaged together as $n_i$ increases.

TABLE I

Casimir Energies Divided by Volume for Massless Scalar ($\varphi$) and Vector ($A$) Fields in $d$ Dimensions Satisfying Periodic ($P$), Dirichlet ($D$), Neumann ($N$), Perfect Conductor ($C$), and "Bag" ($B$) Boundary Conditions on Each Surface of $p$-Dimensional Unit Hypercubes

| $d$ | $p$ | $E_{\varphi P}$ | $E_{\varphi D}$ | $E_{\varphi N}$ | $E_{AC}$ | $E_{AB}$ |
|-----|-----|------|------|------|------|------|
| 1 | 1 | −0.13 | −0.13 | −0.13 | | |
| 2 | 1 | −0.19 | −0.024 | −0.024 | −0.024 | −0.24 |
| 2 | 2 | −0.71 | +0.041 | −0.22 | −0.22 | +0.041 |
| 3 | 1 | −0.11 | −0.0069 | −0.0069 | −0.014 | −0.014 |
| 3 | 2 | −0.31 | +0.0048 | −0.043 | −0.038 | −0.038 |
| 3 | 3 | −0.81 | −0.016 | −0.29 | +0.092 | +0.092 |
| 4 | 1 | −0.085 | −0.0025 | −0.0025 | −0.0075 | −0.0075 |
| 4 | 2 | −0.19 | +0.00081 | −0.013 | −0.011 | −0.025 |
| 4 | 3 | −0.39 | −0.0016 | −0.088 | +0.0096 | −0.047 |
| 4 | 4 | −0.85 | +0.0061 | −0.33 | −0.044 | +0.14 |
| 5 | 1 | −0.065 | −0.0010 | −0.0010 | −0.0041 | −0.0041 |
| 5 | 2 | −0.15 | +0.00012 | −0.0051 | −0.0046 | −0.015 |
| 5 | 3 | −0.27 | −0.00031 | −0.019 | +0.0012 | −0.055 |
| 5 | 4 | −0.48 | +0.00050 | −0.073 | −0.0045 | −0.055 |
| 5 | 5 | −0.95 | −0.0025 | −0.37 | +0.021 | +0.18 |

Some special values of $Z_p(s) \equiv Z_p(a_1 = 1,..., a_p = 1; s)$ are [22]:

$$Z_0(s) = 0,$$

$$Z_1(s) = 2\zeta(s),$$

$$Z_2(s) = 4\zeta\left(\frac{s}{2}\right)\beta\left(\frac{s}{2}\right),$$

$$Z_4(s) = 8(1 - 2^{2-s})\zeta\left(\frac{s}{2}\right)\zeta\left(\frac{s}{2} - 1\right),$$

$$Z_8(s) = 16(1 - 2^{1-s/2} + 2^{4-s})\zeta\left(\frac{s}{2}\right)\zeta\left(\frac{s}{2} - 3\right), \tag{3.5}$$

where $\beta(s) = \sum_{n=0}^{\infty} (-1)^n (2n + 1)^{-s}$ and $\beta(1) = \pi/4$, $\beta(2) = G \simeq 0.915$ ($G$ is Catalan's constant), $\beta(3) = \pi^3/32$. $Z_3(s)$ apparently cannot be expressed as a product of one-dimensional sums.

Figure 3.2 shows the energy divided by volume for $p = 2$ systems as a function of $a_2/a_1$. In all cases, the minimum energy is achieved when $a_1 = a_2$. When $a_2 \gg a_1$ the energy tends exponentially to the result obtained for infinite $a_2$, as mentioned above.
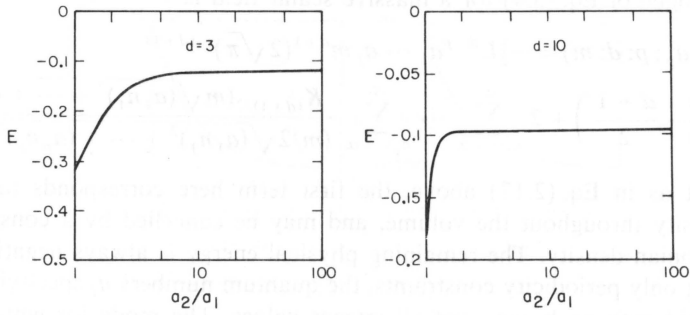
FIG. 3.2. Casimir energy divided by volume for a massless scalar field in $d$ space dimensions constrained to be periodic with periods $a_1$ and $a_2$ along two orthogonal directions.

Figure 3.3 gives contour plots of the energy divided by volume for systems with $p = 3$ in the case $d = 3$ as a function of $a_2/a_1$ and $a_3/a_1$. Again, the minimum energy is achieved in the symmetric configuration $a_1 = a_2 = a_3$. Similar behaviour appears to occur for higher values of $p$. A system of fixed volume containing scalar fields with periodic constraints therefore attains its minimum energy when all $a_i$ are equal.
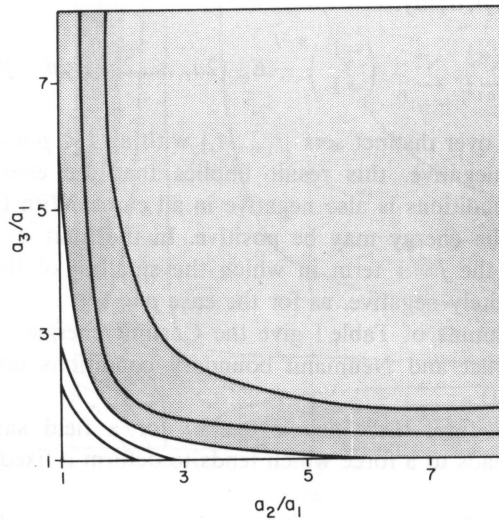


FIG. 3.3. Contour plot for the energy divided by volume of a massless scalar field in three space dimensions constrained to have periods $a_1$, $a_2$ and $a_3$ along the three directions. The energy is always negative. Regions shaded lighter correspond to more negative energies.

The analogue of Eq. (3.4) for a massive scalar field is

$$E_{\varphi_p}(a_1,...,a_p;p;d;m) = -\tfrac{1}{2}L^{d-p}a_1 \cdots a_p m^{d+1}(2\sqrt{\pi})^{-(d+1)} \tag{3.6}$$

$$\times \left[ \Gamma\left(-\frac{d+1}{2}\right) + 2 \sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_p=-\infty}^{\infty} \frac{K_{(d+1)/2}(m\sqrt{(a_1n_1)^2 + \cdots + (a_pn_p)^2})}{(m/2\sqrt{(a_1n_1)^2 + \cdots + (a_pn_p)^2}))^{(d+1)/2}} \right].$$

Notice that as in Eq. (2.17) above, the first term here corresponds to a constant energy density throughout the volume, and may be cancelled by a constant term in the Hamiltonian density. The remaining physical energy is always negative.

Imposing only periodicity constraints, the quantum numbers $n_i$ specifying modes of a scalar field may each run over all integer values. The mode for which all $n_i = 0$ carries zero energy when $m = 0$ and is therefore irrelevant. However, with the Dirichlet boundary conditions (2.2a), each quantum number is restricted to the range $1 \leqslant n_i \leqslant \infty$. Similar, the Neumann condition (2.2b) implies $0 \leqslant n_i \leqslant \infty$. Using the results

$$\sum_{n_1=1}^{N} \cdots \sum_{n_p=1}^{N} f(n_1,...,n_p) = 2^{-p} \sum_{n_1=-N}^{N} [1-\delta_{n_1}] \cdots \sum_{n_p=-N}^{N} [1-\delta_{n_p}] f(n_1,...,n_p),$$

$$\sum_{n_1=0}^{N} \cdots \sum_{n_p=0}^{N} f(n_1,...,n_p) = 2^{-p} \sum_{n_1=-N}^{N} [1+\delta_{n_1}] \cdots \sum_{n_p=-N}^{N} [1+\delta_{n_p}] f(n_1,...,n_p) \tag{3.7}$$

valid for functions $f$ even in the $n_i$, one may write the energies obtained with Dirichlet (D) and Neumann (N) boundary conditions in terms of the result (3.4) as

$$E_{\varphi_{D \atop N}}(a_1,...,a_p;p;d;m)$$

$$= 2^{-p} \sum_{j=1}^{p} \sum_{\{i_1,...,i_j\}} \binom{-1}{+1}^{p-j} E_{\varphi_p}(2a_{i_1},...,2a_{i_j};j;d-p+j;m), \tag{3.8}$$

where the final sum is over distinct sets $\{i_1,...,i_j\}$ with all $i_l \leqslant p$ and $i_1 \neq i_2 \cdots$. Since the $E_{\varphi_p}$ are always negative, this result implies that the energy obtained with Neumann boundary conditions is also negative in all cases. With Dirichlet boundary conditions, however, the energy may be positive. In the limit $d \to \infty$ ($p$ fixed), the sum is dominated by the $j = 1$ term in which the smallest of the $a_i$ appears. The limiting energy is infinitely negative, as for the case $p = 1$.

The second two columns of Table I give the Casimir energies of massless scalar fields satisfying Dirichlet and Neumann boundary conditions on the surface of a hypercube (all $a_i$ equal).

As in the periodic case, the Casimir energy for a field satisfying Neumann boundary conditions leads to a force which tends to deform a fixed volume confining region into a cube.

The case of Dirichlet boundary conditions is more complicated. In the minimum energy shape for a particular value of $d$ and a given volume it appears that some number $r$ of lengths $a_i$ are small and equal, and the rest are large. At least for $2 \leqslant d \leqslant 6$, $r = 1$ when $d$ is even, and $r = d$ when $d$ is odd. When $d$ becomes greater
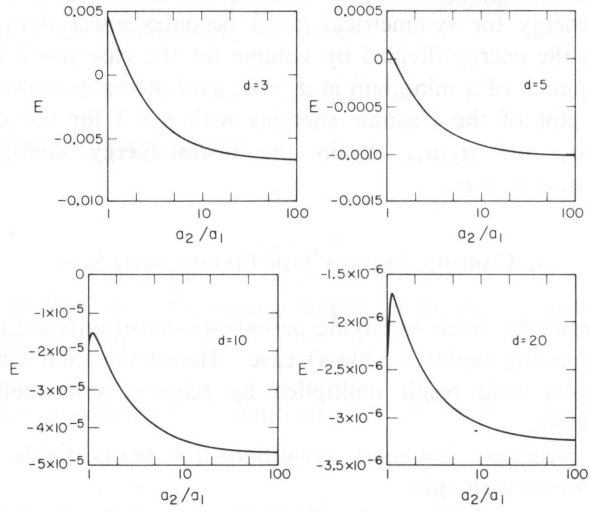
FIG. 3.4. Casimir energy divided by volume for a massless scalar field satisfying Dirichlet boundary conditions on the sides of a rectangular $a_1 \times a_2$ "tube" in $d$ space dimensions.
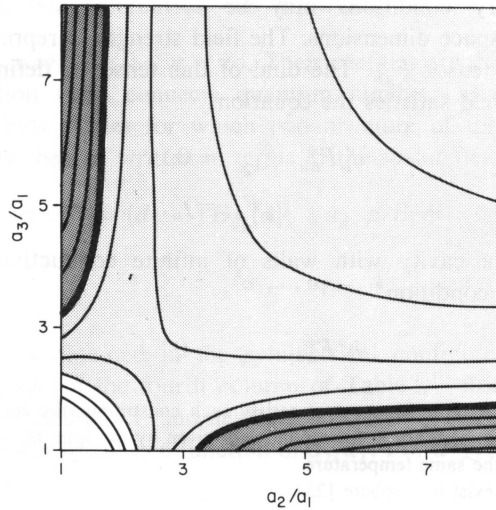
FIG. 3.5. Contour plot for the energy divided by volume of a massless scalar field satisfying Dirichlet boundary conditions on the sides of an $a_1 \times a_2 \times a_{53}$ box in three space dimensions. Regions shaded darker correspond to higher energies. The thick contour is at zero energy.

than about 5.7, the energy for the symmetrical $p = 2$ case ceases to be positive, and for $d \gtrsim 25$, the energy for symmetrical $p = 2$ becomes lower than that for $p = 1$. Figure 3.4 shows the energy divided by volume for the case $p = 2$ as a function of $a_2/a_1$: the development of a minimum at $a_1 = a_2$ as $d$ increases is evident. Figure 3.5 gives a contour plot of the Casimir energies with $p = 3$ for the case $d = 3$ as a function of $a_3/a_1$ and $a_2/a_1$. Again the lowest-energy configuration is the symmetrical one $a_1 = a_2 = a_3$.

## 4. Casimir Energy for Fields with Spin

The mode energies for fields satisfying periodicity constraints are always the same as in the corresponding scalar (spinless) case.[7] Hence the total Casimir energy is given by the scalar field result multiplied by relevant spin multiplicity factors (negative for fermions).

Dirichlet and Neumann boundary conditions for scalar fields have no direct generalization to fields with spin.

The simplest general condition is that the action for a field should vanish outside a specified volume. For massless vector (spin 1) fields, this corresponds to bag boundary conditions, as discussed below. For a massless spinor (spin 1/2) field between two parallel planes this boundary condition yields a Casimir energy: In higher dimensional hyper-cuboids, the presence of corners prevents solutions to the massless Dirac equation with these boundary conditions.[8] Even with $p = 1$, no solutions exist for a massive spinor field [15].

Consistent boundary conditions may be formulated for a massless vector ("photon") field in $d$ space dimensions. The field strength is represented by a totally antisymmetric rank-2 tensor $F_{\mu\nu}$. The dual of this tensor is defined as $F^*_{\mu_1 \cdots \mu_{d-1}} = \varepsilon_{\mu_1 \cdots \mu_{d-1} \nu \lambda} F^{\nu\lambda}$. The field satisfies the equations

$$\partial_\mu F^*_{\mu\nu_1 \cdots \nu_{d-2}} = 0, \tag{4.1a}$$

$$\partial_\mu F_{\mu\nu} = j_\mu. \tag{4.1b}$$

We first consider a cavity with walls of infinite conductivity. The field then satisfies the boundary condition[9]

$$n^\mu F^*_{\mu\nu_1 \cdots \nu_{d-2}} = 0 \tag{4.2}$$

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[7] Fields in equilibrium at finite temperature satisfy such constraints; the energy density of a (non-interacting) field at finite temperature is correspondingly given by the spin multiplicity factor times the result for a scalar field at the same temperature.

[8] Solutions nevertheless exist in a sphere [23].

[9] In the usual case of superconductors in four dimensions, this condition becomes $\mathbf{n} \cdot \mathbf{B} = 0$, $\mathbf{n} \times \mathbf{E} = 0$. In $d$ space dimensions, define a generalized electric field $E_i = F_{0i}$ where $i$ is a space index $1,\ldots, d$. To avoid infinite currents in the conductor, all components of $E_i$ not along $n_\mu$ must vanish. In the conductor, $E_i = 0$ so that $F_{kl} = 0$ for all directions $k$, $l$ orthogonal to $n$. These constraints imply condition (4.2).

on each wall with spacelike normal vector $n^\mu$. In determining modes of the field, it is convenient to introduce potentials $A_\mu$ such that

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad (4.3a)$$

and to work in the "radiation" gauge

$$A_0 = 0, \qquad (4.3b)$$

$$\partial_i A^i = 0.$$

The modes of the field in a conducting hyper-cuboid with $p$ pairs of faces in $d$-dimensional space are then

$$A_i = a_i \cos(k_i x_i) \prod_{\substack{j=1 \\ j \neq i}}^{p} \sin(k_j x_j) \, e^{ik_T x_T - i\omega t} \qquad (1 \leqslant i \leqslant p),$$

$$A_j = ib_j \prod_{l=1}^{p} \sin(k_l x_l) \, e^{ik_T x_T - i\omega t} \qquad (p < j \leqslant d),$$

$$k_i = \pi n_i / a_i,$$

$$\omega = \sqrt{k_i^2 + k_T^2}. \qquad (4.4)$$

where gauge condition (4.3) implies (cf. [24])

$$a_i k_i + \mathbf{b} \cdot \mathbf{k}_T = 0. \qquad (4.5)$$

The modes in (4.4) may have $0 \leqslant n_i < \infty$. Their energies are the same as in the scalar case (2.4). Condition (4.5) connects quantum numbers associated with different directions, and forbids modes for which two or more of the $n_i$ vanish. The total Casimir energy may then be written in terms of the scalar field result (3.4, 3.8) as

$$E_{A_C}(a_1, ..., a_p; p; d; m = 0) = (d - 1) E_{\varphi_D}(a_1, ..., a_p; p; d; m = 0) \qquad (4.6)$$

$$+ \sum_{i=1}^{p} E_{\varphi_D}(a_1, ..., a_{i-1}, a_{i+1}, ..., a_p; p - 1; d - 1; m = 0).$$

The Casimir energies obtained in the symmetrical configuration $a_1 = \cdots = a_p$ for several cases are given in the fourth column of Table I.[10] For low values of $d$, the energy is positive for odd $p > 1$, and negative otherwise. Figure 4.1 shows the energy divided by volume with $p = 2$ as a function of $a_2/a_1$. For $d \lesssim 6$, the minimum energy

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[10] The result for $d = 3$, $p = 2$ agrees with that obtained in [7] with momentum cutoff regularization. The speculation of [13] based on the scalar field calculation that this result should be different in dimensional or zeta function regularization is unfounded: it was implicitly based on incorrect counting of electromagnetic field modes.
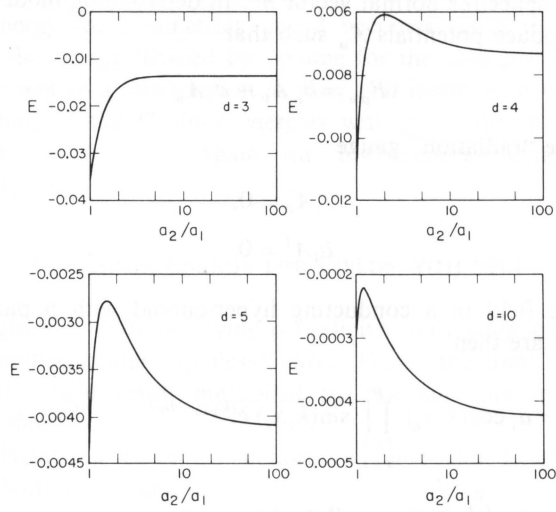
FIG. 4.1.   Casimir energy divided by volume for a massless vector field enclosed in a perfectly conducting rectangular $a_1 \times a_2$ "tube" in $d$ space dimensions.
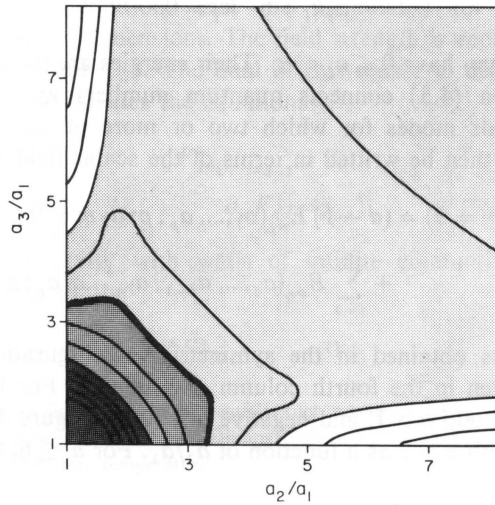


FIG. 4.2.   Contour plot for the energy divided by volume of a massless vector field in a perfectly-conducting $a_1 \times a_2 \times a_3$ box in three space dimensions. Regions shaded darker correspond to higher energies. The thick contour is at zero energy.

is achieved with $a_1 = a_2$; for larger $d$, $a_1 \gg a_2$ gives smaller energy. Figure 4.2 gives a contour plot of the energy divided by volume in the case $d = p = 3$ as a function of $a_2/a_1$ and $a_3/a_1$. When $d = 3$, $a_1 = a_2 = a_3$ gives the maximum energy. The minimum energy is achieved when $a_1 = a_2 \ll a_3$. In three dimensions, Casimir energy should thus tend to deform closed conducting boxes of fixed volume into long tubes. For $d \gtrsim 6$, $a_1 \ll a_2, a_3, \ldots$ is the minimum energy configuration.

An alternative to the perfect conductor boundary condition (4.2) is obtained by requiring the action for the vector field to vanish outside a bounded region. Inserting the necessary step function in the action integral, and varying with respect to the potential $A_\mu$, one obtains at the boundary the constraint[11]

$$n^\mu F_{\mu\nu} = 0, \qquad (4.7)$$

where $n^\mu$ is a spacelike normal vector. This boundary condition is assumed in the bag model for hadrons in QCD. Constraint (4.7) is dual to constraint (4.2). For $d = 3$, this implies identical Casimir energies in the two cases. The modes of a field subject to (4.7) are obtained from (4.4) on interchanging sin and cos. The total energy is given in terms of the scalar field result ((3.4) and (3.8)) by

$$E_{A_B}(a_1, \ldots, a_p; d; p; m = 0)$$

$$= \sum_{j=1}^{p} \sum_{\{i_1, \ldots, i_{p-j+1}\}} (d - j) E_{\varphi_D}(a_{i_1}, \ldots, a_{i_{p-j+1}}; p - j + 1; d - j + 1; m = 0), \qquad (4.8)$$

The last column of Table I lists results for this energy with $a_1 = \cdots = a_p$ in a variety of cases. The energy with $p = 2$ is shown as a function of $a_2/a_1$ in Fig. 4.3. Maximum
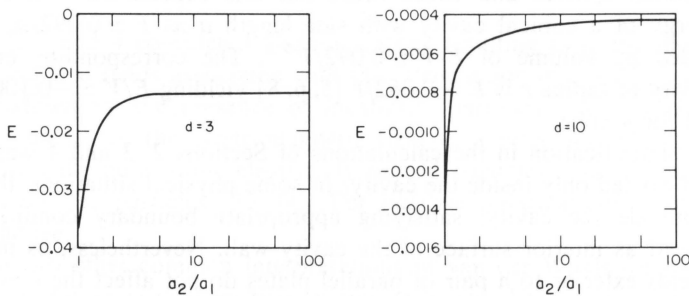


FIG. 4.3. Casimir energy divided by volume for a massless vector field satisfying "bag" boundary conditions on the sides of an $a_1 \times a_2$ "tube" in $d$ space dimensions.

[11] In three dimensions this becomes $\mathbf{n} \times \mathbf{B} = 0$, $\mathbf{n} \cdot \mathbf{E} = 0$, as appropriate at the surface of a material containing infinitely mobile magnetic monopoles.

energy appears to be achieved when $a_1 = \cdots a_p$. The minimum energy for a given volume is attained when $a_1 = \cdots a_{p-1} \ll a_p$.

A massive vector field satisfying the Proca equation in $d$ space dimensions has $d$ degrees of freedom; the corresponding massless field would have $d-1$ degrees of freedom. One may formally impose boundary conditions (4.2) and (4.7) even on a massive vector field. The results are obtained by replacement of the factors $d-1$ and $d-j$ in Eqs. (4.6) and (4.8) with $d$ and $d-j+1$, respectively, and with $m$ equal to the mass of the vector field. The zero mass limits of the corresponding energies do not coincide with the zero mass results obtained above (and given in Table I). However, as shown in [25], Eq. (4.2) no longer gives the boundary condition on a perfect conductor for massive vector fields. The additional (longitudinal) polarization state for massive vector fields is found to decouple in the limit $m \to 0$, and thus presumably contributes no Casimir energy[12] (cf. [26]). The bag boundary condition (4.6), which allows no momentum flux outside a bag, nevertheless maintains its physical significance even for massive vector fields. For $d = 3$, the small mass limit of the Casimir energy with this boundary condition is $-0.021$ for $p = 1$ and $-0.19$ for $p = 3$. These results clearly differ from those for a genuinely massless vector field given in Table I.

For massless spin-2 fields, no physical boundary conditions appear to be consistent with the gauge invariance of the Lagrangian.

## 5. DISCUSSION AND EXTENSIONS

The calculations of Section 3 and 4 were performed only for the case of cuboidal cavities. The Casimir energy is, however, not expected to be sensitive to the detailed shape of the cavity (cf. [6]). Comparison of the electromagnetic Casimir energies for three-dimensional spheres and cubes bears out this expectation. Table I gives the Casimir energy of a cubical cavity with side length $a$ as $E \simeq 0.092/a$, yielding an energy divided by volume of $E/V \simeq 0.092/V^{4/3}$. The corresponding energy for a spherical cavity of radius $r$ is $E \simeq 0.062/r$ [5, 6, 8] yielding $E/V \simeq -0.100/V^{4/3}$, very close to that for a cube.

A further simplification in the calculations of Sections 2, 3 and 4 was to assume that the field existed only inside the cavity. In some physical situations, the field may also exist outside the cavity, satisfying appropriate boundary conditions on the exterior as well as interior surface of the cavity wall. Nevertheless, as mentioned in Section 2, fields exterior to a pair of parallel plates do not affect the Casimir energy associated with the plates. In systems with $p > 1$, however, exterior fields are expected to affect the Casimir energy. Any divergences associated with external mode sums

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[12] The modes of a massive vector field in a cuboidal cavity are complicated when correct boundary conditions are imposed [25]: the sum necessary to obtain the Casimir energy cannot be performed analytically.

should cancel in the computation of physical Casimir energy differences.[13] The contribution of exterior fields for cuboidal regions is difficult to calculate (the necessary modes are discussed in [28]). For a spherical region, inclusion of exterior modes reduces the Casimir energy per unit volume from 0.100 to 0.073 [5, 6, 8]. The energy per unit area for a circle with both interior and exterior modes included is ≃0.040 [29] while the Casimir energy for the interior of a unit square is ≃0.041.

In Sections 2, 3 and 4 we considered only the total Casimir energy of a cavity, and not the energy density as a function of position. Forces exerted on walls confining a field depend only on this total energy. However, gravitational effects may depend on the local energy density.

The energy density for fields with periodicity constraints is always independent of position. For massless fields, this energy density is finite. For massive fields, a single divergence proportional to the total volume appears. As in Section 2, this may be cancelled by addition of a constant to the Hamiltonian density (a "cosmological term" or "bag constant").

In the presence of explicit boundaries the possible divergences and "surface" Lagrangian counterterms become more complicated. For example, for a scalar field the energy-momentum tensor with minimal coupling diverges like $\delta^{-(d+1)}$ when the distance $\delta$ from a boundary tends to zero [30]. With conformal invariant coupling, the divergences in the energy-momentum tensor are proportional to $\delta^{-d}$ times the curvature of the boundary [30–32]. With dimensional regularization, the formal integral of such pure power divergences is zero, and thus does not affect the total Casimir energy. The origin of divergences in the local energy density is presumably the unphysical assumption of a precisely localized boundary. Such a boundary could be maintained only with an infinite binding energy which must compensate the infinite local Casimir energy at the boundary. The binding energy of the boundary depends only on the form of the boundary itself. It is, for example, independent of the separation between two boundary planes, and thus does not contribute to the Casimir force.

Sections 2, 3 and 4 considered Casimir energy resulting from the modification of modes in quantized fields associated with the introduction of boundaries. The modes are also modified by the presence of localized "particles." A polarizable particle perturbs the modes of the electromagnetic field in a large box. The perturbations associated with the polarizable electrically-neutral particles lead to "van der Waals forces" between the particles [33, 34]. For positive polarizability, these forces are always attractive.

There are two other common interpretations of van der Waals forces. In the first (e.g. [35]), the forces arise from two-photon exchange. The corresponding Feynman diagram contains a closed photon loop, which represents vacuum fluctuations in the electromagnetic field, perturbed by two insertions representing the two particles. An integration is performed over the possible momenta of the virtual intermediate

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[13] Finite results require the regularization parameter to be taken independent of the size of the system: if a cutoff on mode number $n$ is used, divergences may remain [27].

photons. In the second interpretation [36, 37], the form (2.5) of the zero-point fluctuations in the electromagnetic field is assumed, and the resulting force of attraction on macroscopic surfaces is calculated.

Given the van der Waals forces between a pair of particles, a simple scheme would take the Casimir forces between surfaces as a superposition of forces between their constituent particles. A conducting surface may be taken to consist of a collection of polarizable particles. According to the simple scheme, the Casimir forces between conducting surfaces would then always be attractive. The results of Section 4 show that in practice they are often repulsive (as in the case of a cubical cavity). The simple scheme fails for essentially two reasons. First, the presence of the boundaries modifies the modes of the electromagnetic field, and thus changes the virtual photon propagator and the spectrum of the zero-point fluctuations in the electromagnetic field. Second, whenever the boudary is connected (as when $p > 1$), it is not possible to separate the "self-energy" of the boundary from the true Casimir energy of the confined field. In the case of two parallel planes, the simple scheme nevertheless leads to a correct Casimir energy. This result is probably fortuitous: the scheme is known to fail for $p > 1$ and probably fails with non-planar boudaries even with $p = 1$.

The failure of superposition for van der Waals forces is in principle amenable to experimental investigation. The repulsive nature of Casimir forces in a spherical cavity could perhaps be seen in the behaviour of small bubbles (possibly in liquid $^4He$).

In physical systems, the assumption of perfectly conducting cavity walls holds at best only for frequencies much lower than those characteristic of the interatomic spacing. The Casimir energies of dielectric systems may be given as integrals over the dielectric constant $\varepsilon(\omega)$ [8, 36, 37]. The decrease in conductivity at high frequencies (leading to free transmission of high modes by the cavity walls) appears to affect only details of Casimir energies: their approximate magnitude and sign is left unchanged.

## 6. APPLICATIONS

We discuss in this section several potential consequences of Casimir energies, mostly for microscopic phenomena.

Sections 3 and 4 showed that the total Casimir energies of finite cavities may be negative. (An example is a sufficiently elongated conducting cavity containing electromagnetic field in three dimensions.) Such negative energies violate the dominant energy condition commonly postulated in classical general relativity (e.g. [38]). Failure of the dominant energy conditions renders the Hawking–Penrose singularity theorems impotent. (Theorems are also invalidated by violations of the strong energy condition $p \leqslant -\rho/3$ [39].) The total energy (or mass) of a cavity is negative only if the energy of its walls is sufficiently small. The calculation of the wall energies can be performed only with an explicit physical model for their construction. One suspects that the result will show that an isolated system with negative total mass cannot

exist.[14] If such a system were possible, then the usual vacuum should be unstable and decay into such systems.[15]

In simple models, composite particles are often taken to consist of a bag containing approximately free fields. An early semiclassical model for the electron involved a charged spherical conductor, whose positive electrostatic energy was cancelled by negative Casimir energy [40, 41]. The requirement of cancellation implied a definite value for the electromagnetic coupling constant $\alpha$. In practice, a spherical (zero angular momentum) conducting cavity or shell has positive [5] rather than negative electromagnetic Casimir energy, and no value of $\alpha$ yields a stable system. The QCD bag model takes hadrons to consist of a bag containing approximately free quark and gluon fields, yielding again a positive Casimir energy [42]. However, in a model with scalar constituents, the results of Section 4 show that binding of a composite particle by Casimir energy is potentially possible.

Casimir energy may also be important in determining the structure of the vacuum state in interacting field theories. According to a simple model (e.g. [43]), the vacuum state in QCD consists of a "foam-like" collection of regions ("bubbles") of low field strength, separated by walls of high field strength. The size of the bubbles is governed by the characteristic distance $\sim 1/\Lambda$ at which the effective QCD coupling constant becomes strong. Similarly, in QED and quantum gravity [44], a foam structure associated witht the increasing strength of the effective coupling at very short distances might be expected. Many phenomena may contribute in determining the structure of the foam. The properties of the walls are crucial, but can presumably be found only by consideration of the complete interacting field theory. The bubbles may contain for example magnetic fields whose energy density determines their size. However, a universal feature is the presence of Casimir energy arising solely from the confinement of the fields in bubbles. Section 4 shows that in, for example, the case of QCD or QED in three space dimensions, this Casimir energy leads to a force which tends to deform bubbles of fixed volume into long tubes. (In a foam, the total volume of each bubble should remain fixed.) If Casimir energy is dominant, this suggests that the vacuum in QCD should consist of a foam of long tubes rather than approximately spherical bags.

For certain dimensionalities and boundary conditions, the Casimir energy is minimized when some of the dimensions of a system are very long compared to the others. In such cases, bubbles of a certain dimensionality should become very thin in some directions and very long in others, so that the dimensionality of their interior is effectively reduced. The bubbles in the vacuum state for an interacting field theory in

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[14] Interactions within a collection of positive and negative classical electric charges contributes a negative energy, but this negative energy is always overwhelmed by positive self energy of the charges. Similarly, the gravitational binding energy of a classical collection of masses is apparently always overwhelmed by kinetic energy, yielding a positive total energy. Similar compensation may occur for quantum mechanical Casimir energy.

[15] The lowest energy state in quantum gravity might nevertheless consist of an assembly of small (almost) closed universes with negative Casimir energies.

$d > 3$ space dimensions could thus deform as a result of Casimir forces into bubbles which are long in, say, 3 directions but short in $d - 3$. Excitations along the three long directions could be approximately massless, but those along the short directions could be unobservable because of their large masses. Casimir energy could thus potentially lead to spontaneous reduction in the effective dimensionality of a field theory.

A complete spatially finite universe also exhibits a non-zero Casimir energy. The resulting pressure may well be anisotropic, as in the case of physical boundaries. An intriguing possibility is that in a high dimensional universe, Casimir energy could cause collapse along some directions, and expansion along others (cf. [45]), thereby reducing the apparent dimensionality of spacetime. A quantitative investigation of this possibility would require evaluation of the Casimir energy in anisotropic universes (e.g., governed by static mixmaster metrics, as recently considered in [46]).

## 7. CASIMIR ENERGIES AT FINITE TEMPERATURE

In this section, we consider free fields in finite volumes, maintained in thermal equilibrium at a temperature $T = 1/\beta$. Introducing a partition function $Z$ for the complete system, one obtains as usual

$$F = \frac{-1}{\beta} \log(Z),$$

$$E = \frac{\partial}{\partial \beta} \log(Z),$$

$$S = -\frac{\partial F}{\partial T} = \beta^2 \frac{\partial}{\partial \beta} \left( \frac{1}{\beta} \log(Z) \right), \qquad (7.1)$$

where $F$ is the Helmholtz free energy, $E$ the total energy and $S$ the entropy.

The partition function $Z$ at finite $\beta$ may be obtained from the path integral expression for the vacuum-to-vacuum amplitude by taking an imaginary time coordinate, in which boson fields have periodicity $\beta$.

We consider first a scalar field with mass $m$, constrained to have periodicities $a_1, ..., a_p$ in $p$ space directions. The field is maintained in thermal equilibrium at a temperature $T = 1/\beta$, and thus has period $\beta$ in the imaginary time direction. Suppressing an irrelevant normalization factor, the vacuum-to-vacuum amplitude is given by

$$e^{-Z} = \int D\varphi \, e^{-\int d^{d+1}x \varphi(-\partial^2 + m^2)\varphi} = \text{Det}[-\partial^2 + m^2]^{-1/2}, \qquad (7.2)$$

where the partition function may be written

$$\log(Z) = -\frac{1}{2} \log \operatorname{Det}(-\partial^2 + m^2) = -\frac{1}{2} \operatorname{Tr} \log(-\partial^2 + m^2)$$

$$= -\frac{1}{2} \sum_{n_0=-\infty}^{\infty} \sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_p=-\infty}^{\infty} \left(\frac{L}{2\pi}\right)^{d-p} \tag{7.3}$$

$$\times \int d^{d-p}k_T \log\left[\left(\frac{2n_0\pi}{\beta}\right)^2 + \left(\frac{2n_1\pi}{a_1}\right)^2 + \cdots + \left(\frac{2n_p\pi}{a_p}\right)^2 + k_T^2 + m^2\right].$$

In the zero temperature limit $\beta \to \infty$ the $n_0$ summation in (4.3) is replaced by an integral over $k_0 = 2\pi n_0/\beta$ and the partition function becomes

$$\log(Z) = -\frac{1}{2}\frac{\beta}{2\pi}\int dk_0 \sum_n \log(k_0^2 + \omega_n^2) = -\frac{1}{2}\beta \sum_n \omega_n, \tag{7.4}$$

where an irrelevant additive constant independent of $a_i$ has been dropped. The corresponding energy is then given by $E = F = \frac{1}{2}\sum \omega_n$, as in Sections 2 and 3 above.

In the case $m = 0$, writing $\log(x) = \lim_{\varepsilon \to 0}(\partial/\partial\varepsilon)x^\varepsilon$, one may use Eqs. (3.2) and (A.13) to obtain

$$\log(Z) = \frac{1}{2}\beta a_1 \cdots a_p L^{d-p}\Gamma\left(\frac{d+1}{2}\right)\pi^{-(d+1)/2}Z_{p+1}(\beta, a_1,..., a_p; d+1). \tag{7.5}$$

A convenient representation at low temperatures (high $\beta$) is obtained using Eq. (A.13):

$$\log(Z) = \frac{1}{2}\beta a_1 \cdots a_p L^{d-p}\Gamma\left(\frac{d+1}{2}\right)\pi^{-(d+1)/2}Z_p(a_1,..., a_p; d+1)$$

$$+ \sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_p=-\infty}^{\infty} \int d^{d-p}k_T \log(1 - e^{-\beta\{k_T^2 + (2\pi n_1/a_1)^2 + \cdots + (2\pi n_p/a_p)^2\}^{1/2}})$$

$$+ \frac{\Gamma(d-p+1)\,\zeta(d-p+1)}{\pi^{(d-p+1)/2}}\frac{L^{d-p}}{\beta^{d-p}} \qquad (d \neq p). \tag{7.6}$$

The corresponding representation for high temperatures (low $\beta$) is obtained from Eq. (A.14) (cf. [47]):

$$\log(Z) = -\frac{1}{2}L^{d-p}a_1 \cdots a_p \Gamma\left(\frac{d}{2}\right)\pi^{-d/2}Z_p(a_1,..., a_p; d)$$

$$- 2L^{d-p}a_1 \cdots a_p \beta^{-d} \sum_{n_1} \cdots \sum_{n_p}' \sum_{n=1}^{\infty} \left[\frac{n\beta}{\sqrt{(a_1 n_1)^2 + \cdots + (a_p n_p)^2}}\right]^{d/2}$$

$$\times K_{d/2}\left(\frac{2\pi n}{\beta}\sqrt{(a_1 n_1)^2 + \cdots + (a_p n_p)^2}\right)$$

$$- L^{d-p}a_1 \cdots a_p \beta^{-d}\Gamma\left(\frac{d+1}{2}\right)\pi^{-(d+1)/2}\zeta(d+1) \qquad (d \neq p). \tag{7.7}$$
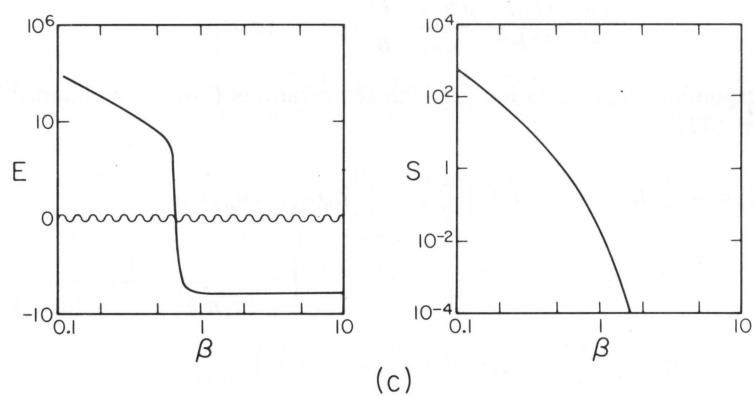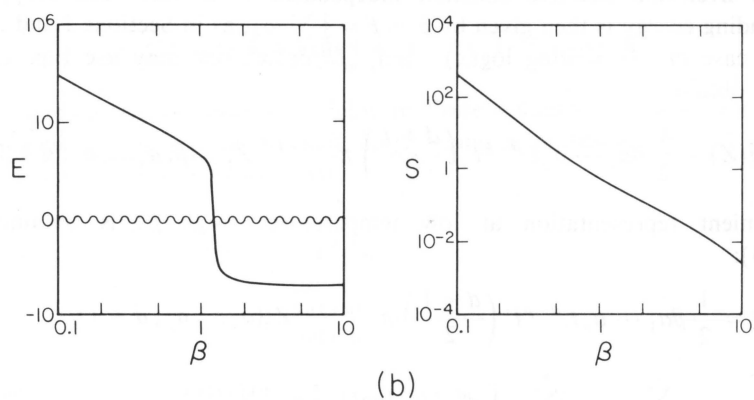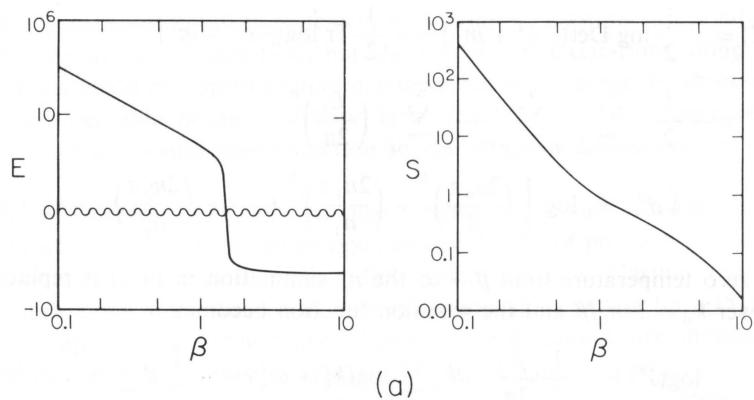
FIG. 7.1.   Energy and entropy as a function of inverse temperature $\beta$ for a massless scalar field in three space dimensions constrained to have unit periods along $p$ orthogonal directions.

When $p = 0$, no constraints are imposed, and Eqs. (7.5), (7.6) and (7.7) yield the standard result for a free Bose gas:

$$E(\beta, p = 0, d) = L^d \beta^{-(d+1)} d\pi^{-(d+1)/2} \zeta(d + 1). \tag{7.8}$$

Figures 7.1a and b show the internal energy and entropy deduced from Eqs. (7.5), (7.6) and (7.7) with $d = 3$ and $p = 1, 2$ using relations (7.1). At low temperatures (high $\beta$), only the first term of Eq. (7.6) survives. The internal energy tends to the zero temperature Casimir value. The entropy tends to zero, reflecting the uniqueness of the vacuum state (third law of thermodynamics). The second and third terms of Eq. (7.6) represent respectively the contributions of discrete modes in the $p$ constrained directions and continuous modes in the $d - p$ unconstrained directions, weighted with the Bose–Einstein distribution at temperature $1/\beta$. The modes in the constrained directions have a non-zero minimum energy, and the second term in (7.6) becomes exponentially small when the temperature falls below this minimum energy ($\beta \sim 1$ in Figs. 7.1a and b. The modes in the unconstrained directions have zero minimum energy, yielding a power law form for the third term. In the high-temperature limit ($\beta \to 0$), only the third term of Eq. (7.7) survives, yielding the Planck result (7.8), independent of $p$.

When $d \to p$, the $\zeta$ function in Eq. (7.5) exhibits a pole. The third term in Eq. (7.6) gives the contribution of modes in the $d - p = 0$ unconstrained directions, and is expected to vanish, but instead formally exhibits a pole. The term is seen to vanish if the contributions of low-frequency modes are regularized by a finite $L$ before the dimensionality is taken to zero. In this case, the partition function for a free Bose gas is given approximately by

$$
\begin{aligned}
&\left(\frac{L}{2\pi}\right) \int_{|k| > 2\pi\kappa/L} d^d k \, \log(1 - e^{-\beta|k|}) \\
&= \frac{2\pi^{d/2}}{\Gamma(d/2 + 1)} \kappa^d \sum_{n=1}^{\infty} \left\{ \left(\frac{L}{2\pi\kappa\beta}\right)^d \frac{\Gamma(d+1, 2\pi\kappa\beta n/L)}{n^{d+1}} - \frac{\Gamma(1, 2\pi\kappa\beta n/L)}{n} \right\},
\end{aligned} \tag{7.9}
$$

where $\Gamma(\alpha, x) = \int_x^\infty t^{\alpha-1} e^{-t} \, dt$ is the incomplete gamma function. For positive integer $d$ (7.9) yields the result (7.8) with corrections of order $(\beta/L)$. For $d = 0$, Eq. (7.8) yields zero in the limit $L \to \infty$. Thus when $d = p$, Eq. (7.6) is valid if the last term is set to zero.

In the high-temperature expansion (7.7) the pole at $d = p$ appears in the first term. It may removed in analogy to the low-temperature case above by subtracting the free Bose gas result (7.8) yielding for the first term

$$
\lim_{d \to p} \frac{1}{2} L^{d-p} \left[ a_1 \cdots a_p \Gamma\left(\frac{d}{2}\right) \pi^{-d/2} Z_p(a_1, ..., a_p; d) \right.
$$

$$
\left. - \Gamma\left(\frac{d-p+1}{2}\right) \pi^{(p-1-d)/2} \beta Z_1(\beta; d-p+1) \right]
$$

$$= \lim_{d \to p} \frac{1}{2} L^{d-p} \Gamma \left( \frac{p-d}{2} \right) \pi^{(p-d)/2} \left[ Z_p \left( \frac{1}{a_1},...,\frac{1}{a_p}; p-d \right) - Z_1 \left( \frac{1}{\beta}; p-d \right) \right]$$

$$= \frac{1}{2} L^{d-p} \Gamma \left( \frac{p-d}{2} \right) \pi^{(p-d)/2} \left[ Z'_d \left( \frac{1}{a_1},...,\frac{1}{a_d}; 0 \right) - Z'_1 \left( \frac{1}{\beta}; 0 \right) \right], \tag{7.10}$$

where $Z'_p(a; s) = (d/ds) Z_p(a; s)$. In the case $a_1 = \cdots = a_p$ this becomes

$$\log \left( \frac{\beta}{a} \right) + \frac{1}{2} \Gamma \left( \frac{d}{2} \right) \pi^{-d/2} Z_{d-1}(a_1 = 1,..., a_p = 1; d)$$

$$+ \sum_{n_1} \cdots \sum_{n_{d-1}} \log(1 - e^{-2\pi \sqrt{n_1^2 + \cdots + n_{d-1}^2}}). \tag{7.11}$$

The resulting internal energy and entropy for $d = p = 3$ are given in Fig. 7.1c. In the high-temperature limit, the first correction to the $\beta^{-4}$ Planck form is $O(\log(\beta))$.

The results for fields at finite temperature obeying periodicity constraints may be extended to fields with other boundary conditions using the relations (3.8) and (4.6), (4.8) just as in the zero temperature case. (In the analogue of Eq. (3.8), however, the sum over $j$ must be extended to include $j = 0$, since for finite $\beta$, $E_{\varphi_p} \neq 0$ when $p = 0$.) With periodic and Neumann boundary conditions, modes with $n_1 = n_2 = \cdots = n_p = 0$ exist, and provide a power correction to the zero temperature Casimir energy at high $\beta$. With Dirichlet, perfect conductor or bag boundary conditions, no such modes can occur, and the corrections are exponential. In the high-temperature limit, corrections to the Planck form with progressively lower powers of temperature are proportional to the areas of progressively smaller subspaces of the boundary. When $d = 3$, the high-temperature result for a vector field obeying either perfect conductor or bag boundary conditions is

$$\log(Z) \simeq \frac{2\zeta(4)}{\pi^2} \frac{a_1 a_2 a_3}{\beta^3} - \frac{\zeta(2)}{2\pi} \frac{a_1 + a_2 + a_3}{\beta} - \frac{1}{2} \log \left( \frac{\beta}{C(a_i)} \right), \tag{7.12}$$

where $C(a_i)$ is a complicated function of the $a_i$ (which could be derived from relations given in [20]).

## 8. ENERGY AND ENTROPY OF FINITE QUANTUM SYSTEMS

In this section, we discuss the results of Section 6 on the energy and entropy of quantized fields in finite volumes at non-zero temperature.

It was recently suggested [2] that the entropy of any finite quantum mechanical system should be bounded by
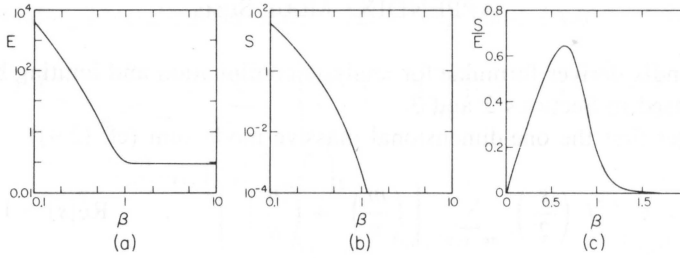
$$S \leqslant 2\pi E R, \tag{8.1}$$

FIG. 8.1. Energy ($E$) and entropy ($S$) of an electromagnetic field in a perfectly conducting three-dimensional cubic cavity. The ratio $S/E$ exhibits a definite maximum.

where $E$ is the energy of the system, and $R$ is the radius of a circumscribing sphere. The equality is realized for a black hole. Bound (8.1) would imply an absolute lower limit on the energy required to transmit information (negentropy) at a particular rate [48].

Since the entropy of a system cannot be negative, the validity of the bound (8.1) requires that $E \geqslant 0$. However, Sections 3 and 4 show that a finite physical system can have $E < 0$.[16] The bound (8.1) is therefore incorrect. Figure 8.1 shows the energy and entropy of an electromagnetic field in a cubical three-dimensional cavity. Notice that the maximum value of $S/E$ is achieved at the point for which $S/E = \beta$, as expected from Ref. [2]. Figure 8.2 shows $S$ and $E$ for a cavity with side lengths in the ratio $1:1:4$, and provides an example of a system for which (8.1) fails.

The existence of a bound such as (8.1) is suggested by the second law of black hole dynamics [2]. This law is however violated by the presence of negative energies through the failure of the dominant energy condition.

Other reasons for and examples of the violation of (8.1) have very recently been given in [49].
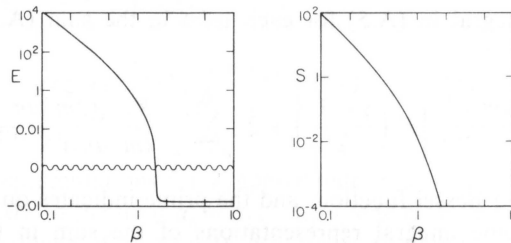


FIG. 8.2. Energy and entropy of an electromagnetic field in a perfectly conducting three-dimensional cavity with sides in the ratio $1:1:4$. The energy is negative for sufficiently low temperatures, and no bound on $S/E$ exists.

---

[16] As discussed in Section 6, however, the "walls" of such systems may nevertheless compensate their negative energy to yield a net positive energy which respects bound (8.1).

## APPENDIX: Mode Sums

This Appendix derives formulae for analytic continuation and limiting behaviour of mode sums used in Sections 2 and 3.

We consider first the one-dimensional massive mode sum (cf. (2.9))

$$S = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \sum_{n=-\infty}^{\infty} \left[\left(\frac{m}{\pi}\right)^2 + \left(\frac{n}{a}\right)^2\right]^{-s/2}, \qquad \text{Re}[s] > 1. \qquad (A.1)$$

We require an analytic continuation of $S$ valid for $s$ negative. Introducing the Jacobi $\vartheta$ function

$$\vartheta(z; x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x} e^{2\pi n z} \qquad (A.2)$$

one may use the integral representation of the gamma function to write

$$S = \int_0^{\infty} dx \, x^{s/2-1} e^{-x(m/\pi)} \vartheta\left(0; \frac{x}{a^2}\right). \qquad (A.3)$$

Applying Jacobi's transformation

$$\vartheta(z; x) = \frac{1}{\sqrt{x}} e^{\pi(z^2/x)} \vartheta\left(\frac{z}{ix}; \frac{1}{x}\right) \qquad (A.4)$$

$S$ becomes

$$S = a\left[\Gamma\left(\frac{s-1}{2}\right)\left(\frac{m}{\sqrt{\pi}}\right)^{1-s} + \int_0^{\infty} dx \, x^{(1-s)/2-1} e^{-m^2/\pi x}[\vartheta(0; a^2 x) - 1]\right]. \quad (A.5)$$

This form provides an analytic continuation for all values of $s$ away from the pole at $s = 1$.

Performing the integral in (A.5) for each term in the sum (A.2) one obtains the large $m$ expansion

$$S = \frac{am^{1-s}}{\pi^{(1-s)/2}}\left[\Gamma\left(\frac{s-1}{2}\right) + 2 \sum_{n=-\infty}^{\infty}{}' \frac{K_{1-s}(2m|an|)}{(m|an|)^{(1-s)/2}}\right], \qquad (A.6)$$

where $K$ is a modified Bessel function, and the prime indicates omission of the $n = 0$ term in the sum. Some integral representations of the sum in (A.6) are given in (2.18).

An analytic continuation for multi-dimensional mode sums is obtained in direct analogy with (A.5) using the generalized Jacobi $\vartheta$ function defined by

$$\vartheta(z_1,..., z_p; x_1,..., x_p) = \prod_{i=1}^{p} \vartheta(z_i; x_i). \qquad (A.7)$$

The corresponding generalization of the large $m$ expansion (A.6) is thus

$$\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_p=-\infty}^{\infty}{}' \left[ \left(\frac{m}{\pi}\right)^2 + \left(\frac{n_1}{a_1}\right)^2 + \cdots + \left(\frac{n_p}{a_p}\right)^2 \right]^{-s/2}$$

$$= \frac{a_1 \cdots a_p m^{p-s}}{\pi^{(p-s)/2}} \left( \Gamma\left(\frac{s-p}{2}\right) \right.$$

$$+ 2 \sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_p=-\infty}^{\infty}{}' \frac{K_{(p-s)/2}\left(2m\sqrt{(a_1 n_1)^2 + \cdots + (a_p n_p)^2}\right)}{\left(m\sqrt{(a_1 n_1)^2 + \cdots + (a_p n_p)^2}\right)^{(p-s)/2}} \right). \quad (A.8)$$

In the massless case $m = 0$, all mode sums may be expressed in terms of the Epstein zeta function (3.2)

$$Z_p(a_1,...,a_p;s) = \sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_p=-\infty}^{\infty}{}' \left[ (n_1 a_1)^2 + \cdots + (n_p a_p)^2 \right]^{-s/2} \quad (s > p), \quad (A.9)$$

An integral representation for this function in terms of the generalized Jacobi $\vartheta$ function is

$$(\xi\pi)^{-s/2} \Gamma\left(\frac{s}{2}\right) Z(a_1,...,a_p;s)$$

$$= -\left(\frac{2}{s} + \frac{2}{p-s}\right) + \xi^{-s/2} \int_{\xi}^{\infty} dx\, x^{s/2-1} (\vartheta_p(0,...,0; a_1^2 x,..., a_p^2 x) - 1)$$

$$+ \xi^{(s-p)/2} \int_{1/\xi}^{\infty} dy\, y^{(p-s)/2-1} \vartheta_p\left(0,...,0; \frac{y}{a_1^2},..., \frac{y}{a_p^2}\right) - 1 \right), \quad (A.10)$$

where $\xi^{p/2} = a_1 \cdots a_p$. This representation gives an analytic continuation for $Z$ except for a pole at $p = s$. The reflection formula (3.4)

$$(\xi\pi)^{-s/2} \Gamma\left(\frac{s}{2}\right) Z_p(a_1,...,a_p;s)$$

$$= (\xi/\pi)^{(p-s)/2} \Gamma\left(\frac{p-s}{2}\right) Z_p(1/a_1,..., 1/a_p; p-s) \quad (A.11)$$

follows.

Several further representations for $Z$ appropriate in different limiting cases may be obtained from (A.10). In the limit $a_p \gg a_1,..., a_{p-1}$ one may sum first over $n_1,..., n_{p-1}$, and then use an analogue of (A.8) with $m = \pi a_p n_p$ to obtain

$$a_1 \cdots a_p \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) Z_p(a_1,...,a_p;s)$$

$$= a_p^{p-s} 2\pi^{(p-s-1)/2} \Gamma\left(\frac{s-p+1}{2}\right) \zeta(s-p+1)$$

$$+ a_p \left[ a_1 \cdots a_{p-1} \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) Z_{p-1}(a_1,...,a_{p-1};s) \right]$$

$$+ \frac{4a_p}{(2\pi)^{(s-p)/2}} \sum_{n_p=1}^{\infty} \sum_{n_1=-\infty}^{\infty} \cdots \sideset{}{'}\sum_{n_{p-1}=-\infty}^{\infty} a_p$$

$$\times \left\{ \left[ \left(\frac{n_1}{a_1}\right)^2 + \cdots + \left(\frac{n_{p-1}}{a_{p-1}}\right)^2 \right] \bigg/ (n_p a_p)^2 \right\}^{(s+1-p)/4}$$

$$\times K_{(p-s-1)/2} \left( 2\pi n_p a_p \left[ \left(\frac{n_1}{a_1}\right)^2 + \cdots + \left(\frac{n_{p-1}}{a_{p-1}}\right)^2 \right]^{1/2} \right)$$

$$= 2a_p^{p-s} \pi^{(p-s-1)/2} \Gamma\left(\frac{s-p+1}{2}\right) \zeta(s-p+1)$$

$$+ 2 \sum_{n_1} \cdots \sideset{}{'}\sum_{n_{p-1}} \int \frac{d^{s-p}k}{(2\pi)^{s-p}} \log(1 - e^{-a_p[k^2 + (2\pi n_1/a_1)^2 + \cdots + (2\pi n_{p-1}/a_{p-1})^2]^{1/2}}).$$

(A.12)

The last term in both forms falls off exponentially with $a_p/a_i$. When $p \to s$, the first term becomes $(2/(s-p))\, a_p^{p-s} - \log 4\pi$, and thus exhibits a pole.

For $a_p \ll a_1,...,a_{p-1}$, a convenient representation is

$$a_1 \cdots a_p \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) Z_p(a_1,...,a_p;s)$$

$$= a_1 \cdots a_p a_p^{-s} 2\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) + a_1 \cdots a_{p-1} \pi^{-(s-1)} \Gamma\left(\frac{s-1}{2}\right)$$

$$\times Z_{p-1}(s-1, a_1,...,a_{p-1})$$

$$+ 4a_p^{-s} \sum_{n=1}^{\infty} \sum_{n_1} \cdots \sideset{}{'}\sum_{n_{p-1}} K_{(s-1)/2} \left( \frac{2\pi n}{a_p} \sqrt{(a_1 n_1)^2 + \cdots + (a_{p-1} n_{p-1})^2} \right)$$

$$\times \left[ \frac{na_p}{\sqrt{(a_1 n_1)^2 + \cdots + (a_{p-1} n_{p-1})^2}} \right]^{(s-1)/2}.$$

## ACKNOWLEDGMENTS

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